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NOTES ON STRONGLY SEMI-CLOSED GRAPH

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Abstract

In these papers, we study on mapping with strongly semi-closed graph. We first introduce the notion "strongly semi-closed graph" in analogue to the closed graph. Also, we study some of their basic properties. This study proved that: If Y is extremally s-disconnected semi-T_2-space and $f: X \to Y$ is a set-s-connected surjection, then $G(f)$ is semi-closed. If Y is Hausdorff space and f is almost semi continuous, then $G(f)$ is strongly semi-closed. If Y is semi-T_2-space and f is irresolute, then $G(f)$ is strongly semi-closed. Semi-closed mapping and semi-closed graph are two separate concepts. If $G(f)$ is a semi-closed and f is surjection(onto), then Y is semi-T_1-space. If the injective S^{**}-open map X is semi**-connected and $G(f)$ is semi-closed then X is semi-T_2-space provided it is T_1-space and locally semi-connected. S^{**}-closed mapping and semi-closed graph are two separate concepts. If Y is semi-regular space, then the following are equivalent $G(f)$ is semi-closed and $G(f)$ is strongly semi-closed

Keywords

Closed graph, semi-closed graph, strongly semi-closed graph

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Introduction. Topologists made generalization for the concept "open sets" because of importance in General Topology and they are now the research topics of many topologists worldwide of which lots of important and interesting results emerged. One of the most well-known notions and also an inspiration source is the notion of semi-open sets introduced by N. Levine [1] in 1963. In 1987 also P.E. Long [2] obtained, among others that the identity function on any space X will have closed graph if and only if X is Hausdorff. In this direction we shall introduce and study some properties of mappings with strongly semi-closed graphs by utilizing semi-open sets and the semi-closure operator. Assume that (X, τ_X) is a topological space and let $A \subseteq X$.

The closure and interior of A are denoted by the letters $\overline{A}$ and A°, respectively. A set A is called semi-open (s-open) [1] in a topological space (X,τ_x) if there exist open set U with $U \subset A \subset \overline{U}$. Alternatively, if $A \subset \overline{A^{\circ}}$. Semi-closed (s-closed) [3] set is the complement of a semi-open set. The letter $\overline{A}^s$ denotes the semi-closure [4] of U which is the intersection of all semi-closed sets containing A.

A subset A is called regular-open (briefly, r-open) if $A = \overline{A}^{\circ}$. A subset A is called regular-closed (briefly, r-closed) if $A = \overline{A^{\circ}}$. The family of all open (resp. closed, s-open, s-closed, r-open, r-closed) sets of A is denoted by $O(X,\tau_x)$ (resp. $C(X,\tau_x)$, $SO(X,\tau_x)$, $SC(X,\tau_x)$, $RO(X,\tau_x)$, $RC(X,\tau_x)$). If $A \in SO(X,\tau_x)$ and $A \in SC(X,\tau_x)$, then A is called semi-closen set [5]. A topological space (X,τ_x) is said to be semi-disconnected [5] if there exist two disjoint nonempty semi-open subsets U and V of X such that $X = U \bigcup V$. The space X is semi-connected [6] if and only if is not disconnected.

Preliminaries. Definition 1 [1, 4–8]. *Let (X,τ_x) and (Y,τ_y) be two topological spaces and $f:(X,\tau_x) \to (Y,\tau_y)$ be a mapping, then*

1) f is called semi-continuous mapping if $f^{(-1)}(A) \in SO(X,\tau_x)$ for every $A \in O(Y,\tau_y)$;

2) f is called irresolute mapping if $f^{(-1)}(A) \in SO(X,\tau_x)$ for every $A \in SO(Y,\tau_y)$;

3) f is called almost continuous mapping if $f^{-1}(A) \in O(X,\tau_x)$ for every $A \in RO(Y,\tau_y)$;

4) f is called almost continuous mapping if $f^{-1}(A) \in C(X,\tau_x)$ for every $A \in RC(Y,\tau_y)$;

5) f is semi-open if $f(A) \in SO(Y,\tau_y)$ for every $A \in O(X,\tau_x)$;

6) f is semi-closed if $f(A) \in SC(Y,\tau_y)$ for every $A \in C(X,\tau_x)$;

7) f is S^{**}-open if $f(A) \in SO(Y,\tau_y)$ for every $A \in SO(X,\tau_x)$;

8) f is S^{**}-closed if $f(A) \in SC(Y,\tau_y)$ for every $A \in SC(X,\tau_x)$;

*9) f is semi**-connected if the image every semi-connected set in X is semi-connected set in Y;*

10) f is set-s-connected if and only if for every semi-closen subset F of $f(x)$, $f^{(-1)}(F)$ is semi-closen in X.

Definition 2 [1]. *Let* (X, τ_x) *and* (Y, τ_y) *be two topological spaces and* $f:(X, \tau_x) \to (Y, \tau_y)$ *be a mapping, then* f *is called semi-continuous if for each* $x \in X,$ *and each* $V \in O(X, f(x)),$ *there exists* $U \in SO(X, x)$ *such that* $f(U) \subseteq V.$

Definition 3 [7, 10]. *Let* (X, τ_x) *and* (Y, τ_y) *be two topological spaces and* $f:(X, \tau_x) \to (Y, \tau_y)$ *be a mapping, then* f *is called:*

1) almost semi-continuous if for each $x \in X$ *and each* $V \in RO(Y, f(x)),$ *there exists a* $U \in SO(X, x)$ *such that* $f(U) \subset V;$

2) semi-irresolute if for each $x \in X,$ *and each* $V \in SO(Y, f(x)),$ *there exists* $U \in SO(X, x)$ *such that* $f(U) \subset V.$

Definition 4 [5]. *A topological space is said to be locally semi-connected if for each* $x \in X,$ *and each* $U \in SO(X, x)$ *there exists* $V \in SO(X, x)$ *such that* $x \in V \subset U,$ *where* V *is semi-connected.*

Definition 5 [5]. *A topological space* (X, τ_x) *is said to be extremely s-disconnected if the semi-closure of every semi-open in* X *is semi-open.*

Lemma 1. *In a topological space if* E *be a semi-connected set and* F *be any other set such that* $E \subset F \subset \overline{E}^s,$ *then* F *is semi-connected.*

Definition 6 [7]. *Assume that* (X, τ_x) *is a topological space. Then* X *is semi-*T_1*-space if for each* $a, b \in X$ *such that* $a \neq b,$ *there exists a semi-open set a* W *of* X *containing* a *but not* b *and a semi-open set* U *of* X *containing* b *but not* $a.$

Definition 7 [7]. *Assume that* (X, τ_x) *is a topological space. Then* X *is semi-*T_2*-space if for every* $a, b \in X$ *with* $a \neq b.$ *There exists* $U, W \in SO(X, \tau_x)$ *with* $a \in W$ *and* $b \in U$ *such that* $W \cap U = \varnothing.$

Definition 8. *Assume that* (X, τ_x) *be a topological space, then* X *is semi-regular (semi-R) if for each* $F \in SC(X, \tau_x)$ *with* $x \notin F,$ *then there exists* $W_1, W_2 \in SO(X, \tau_x)$ *such that* $x \in W_1,$ $F \subseteq W_2$ and $W_1 \cap W_2 = \varnothing.$

Semi-closed graph. Definition 9 [11]. *Assume that* X *and* Y *are topological spaces. Let* $f:(X, \tau_x) \to (Y, \tau_y)$ *be any mapping, the graph of* f *is defined as the subset* $G(f) = \{(x, f(x)) : x \in X\}$ *of the product space* $(X \times Y,\ \tau_x \times \tau_y)$ *and is denoted by* $G(f).$

Lemma 2 [12]. *Assume that* X *and* Y *are topological spaces. Let* $f:(X, \tau_x) \to (Y, \tau_y)$ *be any mapping. Then* $G(f)$ *is closed if for every* $x \in X,$ $y \in Y$ *such that* $y \neq f(x),$ *there exist* $U \in O(X, x),$ $V \in O(Y, y)$ *with* $f(U) \cap V = \varnothing.$

Definition 10 [13]. *Assume that* X, Y *be topological spaces. Let* $f:(X,\tau_x)\to(Y,\tau_y)$ *be a mapping, if for each* $(x,y)\in X\times Y-G(f)$, *there exists* $U\in SO(X,x)$, $W\in SO(Y,y)$ *with* $[U\times W]\bigcap G(f)=\varnothing$, *then* $G(f)$ *is semi-closed.*

Example 1. Let $X=\{a,b,c,d\}$, $\tau_x=\{\varnothing,X,\{a\},\{c,d\},\{a,c,d\}\}$ be a topology defined on X, $Y=\{a,b,c\}$, $\tau_y=D$ (briefly discreet topology) be a topology defined on Y, $SO(X,\tau_x)=\{\varnothing,X,\{a\},\{a,b\},\{c,d\}\{a,c,d\},\{b,c,d\}\}$ $SO(Y,\tau_y)=D$. Let $f:(X,\tau_x)\to(Y,\tau_y)$ be a mapping defined by $f(c)=f(d)=c$, $f(a)=a$, $f(b)=b$. Observe, that $G(f)$ is semi-closed.

Lemma 3 [13]. *Assume that* X, Y *be topological spaces. Let* $f:(X,\tau_x)\to\to(Y,\tau_y)$ *be a mapping, if for each* $(x,y)\in X\times Y-G(f)$, *there exists* $U\in SO(X,x)$, $W\in SO(Y,y)$ *with* $f(U)\bigcap W=\varnothing$, *then* $G(f)$ *is semi-closed.*

Proposition 1 [13]. *It is clear that every closed graph is semi-closed.*

The following example shows that the converse of proposition 1 is not true.

Example 2. Let $X=\{a,b\}$, $\tau_x=D$ be a topology defined on X; $Y=\{a,b,c,d\}$, $\tau_y=\{\varnothing,\{c,d\},Y\}$ be a topology defined on Y with $SO(X,\tau_x)=D$ and $SO(Y,\tau_y)=\{\varnothing,Y,\{c,d\},\{a,c,d\},\{b,c,d\}\}$.

Let $f:(X,\tau_x)\to(Y,\tau_y)$, defined by $f(a)=a$, $f(b)=b$. Then $G(f)$ is semi-closed but not closed.

Remark 1. *As illustrated in the example below, a semi-closed mapping need not have a semi-closed graph.*

Example 3. Let $X=Y=\{a,b,c\}$, $\tau_x=\{\varnothing,X,\{a\},\{a,b\}\}$ be a topology defined on X; $\tau_y=\{\varnothing,Y,\{a\}\}$, be a topology defined on Y: $SO(X,\tau_x)=\{\varnothing,X,\{a\},\{a,b\},\{a,c\}\}$, $SO(Y,\tau_y)=\{\varnothing,Y,\{a\},\{a,b\},\{a,c\}\}$ $SC(X,\tau_x)=\{\varnothing,X,\{b,c\},\{c\},\{b\}\}$, and $SC(Y,\tau_y)=\{\varnothing,Y,\{b,c\},\{c\},\{b\}\}$.

Let $f:(X,\tau_x)\to(Y,\tau_y)$ be a mapping defined by $f(a)=a$, $f(b)=f(c)=b$. Observe that f is a semi-closed but $G(f)$ is not semi-closed.

Remark 2. *As shown in the following example, mappings having a semi-closed graph need not be semi-closed mapping.*

Example 4. Let $X=Y=\{a,b,c,d\}$, $\tau_x=D$ be a topology defined on X; $\tau_y=\{\varnothing,Y,\{a\},\{b\},\{a,b\}\}$, be a topology defined on Y; $SO(X,\tau_x)=D$,

$SO(Y,\tau_y)=\{\varnothing, Y,\{a\},\{b\},\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{a,b,c\},\{a,b,d\},$ $\{a,c,d\}, \{b,c,d\}\}$.

Let $f:(X,\tau_x)\to(Y,\tau_y)$ be a mapping defined by $f(a)=a$, $f(b)=b$, $f(c)=c$, $f(d)=d$. Observe that $G(f)$ is a semi-closed but f is not semi-closed.

Remark 3. *Semi-closed mapping and semi-closed graph are two separate concepts. S**-closed mapping and semi-closed graph are two separate concepts.*

Theorem 1. *If $G(f)$ is a semi-closed and $f:(X,\tau_x)\to(Y,\tau_y)$ is surjection(onto). Then Y is semi-T_1-space.*

◄ Assume that $y_1, y_2\in Y$, where $y_1\neq y_2$. We will to show that Y is semi-T_1-space. Since f surjective, there exists $x_0\in X$ such that $f(x_0)=y_2$. Let $(x_0,y_1)\in X\times Y-G(f)$. Since $G(f)$ is semi-closed given, there exists $U_1\in SO(X,x_0)$, $V_1\in SO(Y,y_1)$ with $f(U_1)\bigcap V_1=\varnothing$ (Lemma 3). Now $x_0\in U_1$ implies $f(x_0)=y_2\in f(U_1)$. From the fact $f(U_1)\bigcap V_1=\varnothing$. We get $y_2\notin V_1$. Once more, since f surjective, there exists $x_1\in X$ with $f(x_1)=y_1$. Let $(x_1,y_2)\in X\times Y-G(f)$. Since $G(f)$ is semi-closed given, there exists $U_2\in SO(X,x_1)$, $V_2\in SO(Y,y_2)$ with $f(U_2)\bigcap V_2=\varnothing$ by (Lemma 3). Now $x_1\in U_2$ implies $f(x_1)=y_1\in f(U_2)$. We get $y_1\notin V_2$. Furthermore, it $V_1, V_2\in SO(Y,\tau_y)$ such that $y_1\in V_1$, but $y_2\notin V_1$, $y_2\in V_2$, but $y_1\notin V_2$. Observe that Y is semi-T_1-space. ►

Theorem 2. *If $G(f)$ is semi-closed and $f:(X,\tau_x)\to(Y,\tau_y)$ is injective (one-one). Then X is semi-T_1-space.*

◄ Assume that $x_1, x_2\in X$ where $x_1\neq x_2$ we will to show that X is semi-T_1-space. Since f injective, implies $f(x_1)\neq f(x_2)$. Let $(x_1, f(x_2))\in$ $\in X\times Y-G(f)$. Since $G(f)$ is semi-closed given, by Lemma 3 there exists $U\in SO(X,x_1)$, $V\in SO(Y,f(x_2))$ with $f(U)\bigcap V=\varnothing$. Therefore, $(x_2, f(x_1))\in X\times Y-G(f)$. Once more, $x_2\notin U$, then $f(x_2)\notin f(U)$. Since $G(f)$ is semi-closed given, by Lemma 3, there exists $A\in SO(X,x_2)$, $B\in SO(Y,f(x_1))$ with $f(A)\bigcap B=\varnothing$. Therefore, $x_1\notin A$, then $f(x_1)\notin$ $\notin f(A)$. Furthermore, it observe $x_1\notin A$, but $x_2\in A$, and $x_2\notin U$, but $x_1\in U$, that such $U, A\in SO(X,\tau_x)$. Observe that X is semi-T_1-space. ►

Corollary 1. *If $f:(X,\tau_x)\to(Y,\tau_y)$ is be injective (one-one, onto) and $G(f)$ is semi-closed, then X and Y are both semi-T_1-space.*

◄ Theorems 1 and 2. ►

Theorem 3. *Let* $f:(X,\tau_x)\to(Y,\tau_y)$ *be injective, irresolute (S**-continuous) and* $G(f)$ *is semi-closed, then* X *is semi-*T_2*-space.*

◄ Assume that $x_1, x_2 \in X$, where $x_1 \neq x_2$. Since f injective, implies $f(x_1)\neq f(x_2)$. Let $(x_1, f(x_2))\in X\times Y - G(f)$. Since $G(f)$ is semi-closed given, there exists $U\in SO(X,x_1)$, $V\in SO(Y,f(x_2))$ with $f(U)\bigcap V=\varnothing$ (Lemma 3). Whence, one obtains $U\bigcap f^{-1}(V)=\varnothing$.

So, since f S**-continuous we get $f^{-1}(V)\in SO(X,x_2)$. Furthermore it $U, f^{-1}(V)\in SO(X,\tau_x)$ such that $x_1\in U$, but $x_2\notin U$, and $x_2\in f^{-1}(V)$, but $x_1\notin f^{-1}(V)$ such that $U\bigcap f^{-1}(V)=\varnothing$. Observe that X is semi-T_2-space. ►

Theorem 4. *Let* $f:(X,\tau_x)\to(Y,\tau_y)$ *be any S**-open surjection and* $G(f)$ *is semi-closed. Then* Y *is semi-*T_2*-space.*

◄ Assume that $y_1, y_2\in Y$, where $y_1\neq y_2$. Since f surjective, there exists $x\in X$ with $f(x)=y_2$. Now $(x,y_1)\in X\times Y-G(f)$. Since $G(f)$ is semi-closed given, there exists $U\in SO(X,x)$, $V\in SO(Y,y_1)$ with $f(U)\bigcap V=\varnothing$ (Lemma 3). Since f S**-open, we get $f(U)\in SO(Y,\tau_y)$.

Also, $x\in U$ implies $f(x)=y_2\in f(U)$. Consequently, $V, f(U)\in$ $\in SO(Y,\tau_y)$ with $y_1\in V$, but $y_2\notin V$, and $y_2\in f(U)$, but $y_1\notin f(U)$ such that $f(U)\bigcap V=\varnothing$. Observe that Y is semi-T_2-space. ►

Corollary 2. *If* $f:(X,\tau_x)\to(Y,\tau_y)$ *is bijective, S**-continuous, S**-open and* $G(f)$ *is semi-closed, then* X *and* Y *are both semi-*T_2*-space.*

◄ Theorems 3 and 4. ►

Theorem 5. *If the injective S**-open map-semi** is* $f:(X,\tau_x)\to(Y,\tau_y)$ *connected, and* $G(f)$ *is semi-closed, then* X *is semi-*T_2*-space provided it is* T_1*-space and locally semi-connected.*

◄ Assume that $x_1, x_2\in X$, where $x_1\neq x_2$. Since f injective, implies $f(x_1)\neq f(x_2)$. Now $(x_1, f(x_2))\in X\times Y-G(f)$. Since $G(f)$ semi-closed given, there exists $U_1\in SO(X,x_1)$, $V\in SO(Y,f(x_2))$ with $f(U_1)\bigcap V=\varnothing$ (Lemma 3). Since X is local semi-connected at x_1, there exists $U\in SO(X,x_1)$, where U is semi-connected such that $x_1\in U\subset U_1$. Furthermore, it $f(U)\bigcap V=\varnothing$. Since f is S**-open, $f(U)\in SO(Y,\tau_y)$. We assert that $x_2\notin \bar{U}^s$. Suppose not. We shall now show that $U\bigcup\{x_2\}$ is semi-connected. Since X is T_1-space implies $\{x_2\}$ is a closed set. Thus $U\subset U\bigcup\{x_2\}\subset$

$\subset \overline{(U \bigcup \{x_2\})}^s = \bar{U}^s \bigcup \{x_2\} = \bar{U}^s$. By Lemma 1, $U \bigcup \{x_2\}$ is semi-connected. Since f is semi**-connected $f[U \bigcup \{x_2\}] = f(U) \bigcup \{f(x_2)\}$ is semi-connected. As a consequence, there is an irony. Because $f(U), V \in SO(Y, \tau_y)$, with $f(U) \bigcap V = \varnothing$. So, $x_2 \notin \bar{U}^s$. Setting $U_0 = X - \bar{U}^s$ we get $U \in SO(X, x_1)$ and $U_0 \in SO(X, x_2)$ with $U \bigcap U_0 = \varnothing$. As a consequence, X is semi-T_2-space.

Theorem 6. *If Y is extremally s-disconnected semi-T_2-space and $f:(X, \tau_x) \to (Y, \tau_y)$ is a set-s-connected surjection. Then $G(f)$ is semi-closed.*

◄ Assume, that $G(f)$, where $y \neq f(x)$. Since Y semi-T_2-space, there exists $W \in SO(Y, y)$ with $f(x) \notin \bar{W}^s = V$. Since Y is extremally *s*-disconnected, then V is semi-closen in Y and $f(x) \notin V$. Again since f is set-*s*-connected surjection $f^{-1}(V)$ is semi-closen in X and $x \notin f^{-1}(V)$. Let $U = X - f^{-1}(V)$, then $U \in SO(X, x)$, $V \in SO(Y, y)$ such that $f(U) \bigcap V = \varnothing$. Observe, that $G(f)$ is semi-closed. ►

Strongly semi-closed graph. Definition 11 [12]. *Assume that X and Y be topological spaces. If for each $(x, y) \in X \times Y - G(f)$, there exist $U \in O(X, x)$, $W \in O(Y, y)$ such that $[\mathrm{U} \times \bar{W}] \bigcap G(f) = \varnothing$. Then $G(f)$ is strongly closed.*

Lemma 4 [12]. *Let $f:(X, \tau_x) \to (Y, \tau_y)$ be any mapping, if for each $x \in X$, $y \in Y$, $y \neq f(x)$, there exists $U \in O(X, x)$, $W \in O(Y, y)$ such that $f(U) \bigcap \bar{W} = \varnothing$. Then $G(f)$ is strongly closed.*

Remark 4 [13]. *As shown following example, a mapping having a strongly closed graph need not be almost continuous.*

Example 5 [12]. Let X be the closed unit interval with the usual subspace topology and let Y be the closed unit interval with the topology generated by the subspace topology together with the set $A = \{r : r$ is rational and $A = \{r : r$ is $1/3 < r < 2/3\}$. The identity function $i: X \to Y$ has a closed graph and, moreover, the graph is strongly-closed. Note that i is not almost continuous.

Remark 5. *As illustrated in the example below, almost continuous mapping need not be having a strongly closed graph.*

Example 6. Let $X = \{1,2,3,4\}, \tau_x = \{\varnothing, X, \{3\}\}$ be a topology defined on X; $Y = \{5,6,7\}$, $\tau_y = \{\varnothing, Y, \{5\}\}$ be a topology defined on Y. Let $f:(X, \tau_x) \to (Y, \tau_y)$, defined by: $f(1) = f(2) = 5$, $f(3) = 6$ and $f(4) = 7$. Hence f is almost continuous, but $G(f)$ is not strongly closed.

Remark 6. *Almost continuous mapping and strongly closed graph are two separate concepts.*

Definition 12 [14]. *Assume that* X *and* Y *be topological spaces. Let* $f:(X,\tau_x)\to(Y,\tau_y)$ *be a mapping, if for each* $(x,y)\in X\times Y-G(f)$, *there exist* $U\in SO(X,x)$, $W\in SO(Y,y)$ *with* $[U\times\overline{W}^s]\bigcap G(f)=\varnothing$, *then* $G(f)$ *is strongly semi-closed.*

Example 7. Let $X=Y=\{a,b,c,d\}$, $\tau_x=D$, be a topology defined on X, $\tau_y=\{\varnothing,\{a\},\{b\},\{ab\},Y\}$ be a topology defined on Y, $SO(X,\tau_x)=$ $=D$, $SO(Y,\tau_y)=\{\varnothing,Y,\{a\},\{b\},\{a,b\},\{a,c\},\{a,d\},\{b,c\}\}$, Let $f:(X,\tau_x)\to$ $\to(Y,\tau_y)$ defined by $f(a)=a$, $f(d)=d$, $f(b)=b$, $f(c)=c$. Then $G(f)$ is strongly semi-closed.

Lemma 5 [14]. *Assume that* X *and* Y *be topological spaces. Let* $f:(X,\tau_x)\to(Y,\tau_y)$ *be a mapping, if for each* $(x,y)\in X\times Y-G(f)$, *there exists* $U\in SO(X,x)$, $W\in SO(Y,y)$ *such that* $f(U)\bigcap\overline{W}^s=\varnothing$, *then* $G(f)$ *is strongly semi-closed.*

Proposition 2 [14]. *Any strongly semi-closed graph is semi-closed.*

The converse of proposition 2 is not necessarily true as shown by the following example.

Example 8. Let $X=\{a,b\}$, $\tau_x=\{\varnothing,X,\{a\},\{b\}\}$ be a topology defined on X, $Y=\{a,b,c\}$, $\tau_y=\{\varnothing,Y,\{c\},\{a,c\},\{b,c\}\}$ be a topology defined on Y with $SO(X,\tau_x)=\tau_x$ and $SO(Y,\tau_y)=\tau_y$. Let $f:(X,\tau_x)\to(Y,\tau_y)$, defined by $f(a)=a$, $f(b)=b$. Then $G(f)$ is semi-closed but not strongly semi-closed.

Theorem 7. *Let* $f:(X,\tau_x)\to(Y,\tau_y)$ *be a mapping, if* $G(f)$ *is strongly semi-closed, then for each* $x\in X$, $y\in\bigcap\left\{\overline{f(U)}^s:U\in SO(X,x)\right\}$.

◄ Assume the theorem is incorrect. Then there exists $y\neq f(x)$ such that $y\in\bigcap\left\{\overline{f(U)}^s:U\in SO(X,x)\right\}$. This means $y\in\overline{f(U)}^s$ for each $U\in SO(X,x)$. So, $V\bigcap f(U)\neq\varnothing$ for every $V\in SO(Y,y)$. As a result, it can be concluded that $\overline{V}^s\bigcap f(U)\supset V\bigcap f(U)\neq\varnothing$. This contradicts the hypothesis that $G(f)$ is a strongly semi-closed. As a result, the theorem is true. ►

Theorem 8. *Let* $f:(X,\tau_x)\to(Y,\tau_y)$ *be any mapping. If* Y *is semi-*$\mathcal{R}$*-space, then the following are equivalent:*

1) $G(f)$ *is semi-closed;*
2) $G(f)$ *is strongly semi-closed.*

◄ Assume that 1) hold, to proof $G(f)$ is strongly semi-closed. Let $(x,y) \in X \times Y - G(f)$, where $y \neq f(x)$. Since $G(f)$ semi-closed, there exists $U \in SO(X,x), V \in SO(Y,y)$ such that $f(U) \cap V = \varnothing$. Since Y is semi-$\mathcal{R}$-space, there exists $W \in SO(Y,y)$ with $\overline{W}^s \subseteq V$. Therefore, $f(U) \cap \overline{W}^s = \varnothing$. Observe, that $G(f)$ is strongly semi-closed.

Assume that 2) hold, to proof $G(f)$ is semi-closed. Let $y \neq f(x)$. Then $G(f)$ is strongly semi-closed, there exists $U \in SO(X,x)$ and $V \in SO(Y,y)$ such that $f(U) \cap \overline{V}^s = \varnothing$. Since $V \subset \overline{V}^s$, implies $f(U) \cap V = \varnothing$. Observe, that $G(f)$ is semi-closed. ►

Proposition 3. *A mapping has a strongly semi-closed graph need not be a has closed graph.*

Example 9. In Example 7 clearly $G(f)$ is strongly semi-closed, but not a closed.

Proposition 4. *A mapping has a closed graph need not be a has strongly semi-closed graph.*

Example 10. In example (10) clearly $G(f)$ is closed, but not a strongly semi-closed.

Theorem 9. *If Y is Hausdorff space and $f:(X,\tau_x) \to (Y,\tau_y)$ is almost semi-continuous. Then $G(f)$ is strongly semi-closed.*

◄ Assume that $(x,y) \in X \times Y - G(f)$, where $y \neq f(x)$. Since Y is Hausdorff space, there exists $V \in O(Y,y)$ such that $f(x) \notin \overline{V}$. Hence $\overline{V} \in RC(Y,\tau_y)$. Therefore, $Y - \overline{V} \in RO(Y, f(x))$. Since f is almost semi-continuous, there exists $U \in SO(X,x)$ with $f(U) \subset Y - \overline{V}$ whence $f(U) \cap \overline{V} = \varnothing$. Since $\overline{V}^s \subset \overline{V}$. Hence, $f(U) \cap \overline{V}^s = \varnothing$. Observe, that $G(f)$ is strongly semi-closed. ►

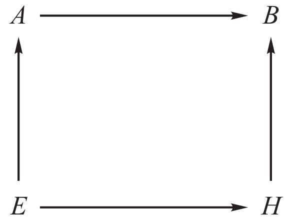
Corollary 3. *If Y is Haudforff space and $f:(X,\tau_x) \to (Y,\tau_y)$ is semi-continuous. Then $G(f)$ is strongly semi-closed.*

◄ Since semi-continuity implies almost semi-continuity, the result follows. ►

Theorem 10. *If Y is semi-T_2-space and $f:(X,\tau_x) \to (Y,\tau_y)$ is irresolute. Then $G(f)$ is strongly semi-closed.*

◄ Assume that $(x,y) \in X \times Y - G(f)$, where $y \neq f(x)$. Since Y is semi-T_2-space, there exists $V \in SO(Y,y)$ with $f(x) \notin \overline{V}^s$. Then $Y - \overline{V}^s \in$ $\in SO(Y, f(x))$. Since f is irresolute, $U \in SO(X,x)$ such that $f(U) \subset$ $\subset Y - \overline{V}^s$. Consequently, $f(U) \cap \overline{V}^s = \varnothing$. Observe, that $G(f)$ is strongly semi-closed. ►

Example 11. Let $X=\{a,b,c,d\}$, $\tau_x=\{\varnothing, X,\{c,d\}\}$ be a topology defined on X, $SO(X,\tau_x)=\{\varnothing, X,\{c,d\},\{a,c,d\},\{b,c,d\}\}$ and let $f:(X,\tau_x)\to$ $\to(X,\tau_x)$ the identity mapping. Then f clearly irresolute and X is not semi-T_2-space. Hence, $G(f)$ is not strongly semi-closed.



The relationship between closed graph, semi-closed graph, strongly-closed graph and strongly semi-closed graph

Theorem 11. *If* $f:(X,\tau_x)\to(Y,\tau_y)$ *is surjective and* $G(f)$ *is strongly semi-closed, then* Y *is semi-*T_2*-space.*

◄ Assume, that $y_1, y_2\in Y$, where $y_1\neq y_2$. Since f is surjective, there exists $x_1\in X$ with $f(x_1)=y_1$. Let $(x_1,y_2)\in X\times Y-G(f)$. Since $G(f)$ is strongly semi-closed, there exists $U\in SO(X,x_1), V\in SO(Y,y_2)$ with $f(U)\bigcap\overline{V}^s=\varnothing$. Consequently, $y_1\notin\overline{V}^s$. Let $W=Y-\overline{V}^s$ and $y_1\in W$. We get, $W\in SO(Y,y_1)$, $V\in SO(Y,y_2)$ with $W\bigcap\overline{V}^s=\varnothing$. Since $V\subseteq\overline{V}^s$, implies $W\bigcap V=\varnothing$. Hence, Y is semi-T_2-space. ►

Remark 7. *The following diagram (figure) shows the relation between* $A=$ *closed graph,* $B=$*semi-closed graph,* $E=$*strongly-closed graph,* $H=$*strongly semi-closed graph.*

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