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# AXISYMMETRIC STRESSED STATE OF UNIFORMLY LAYERED SPACE WITH PERIODIC SYSTEMS OF INTERNAL DISC-SHAPED CRACKS AND INCLUSIONS

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**Abstract**

Using the Hankel integral transform, we construct discontinuous solutions for the problem of the axisymmetric stress state of a piecewise homogeneous, uniformly layered space, obtained by alternately connecting two heterogeneous layers of the same thickness. The space on the middle planes of the first heterogeneous layer contains a periodic system of circular disc-shaped parallel cracks, and on the middle planes of the second layer has a periodic system of circular disc-shaped parallel rigid inclusions. The determining system of equations is obtained in the form of a system of integral equations with kernels of the Weber — Sonin type with respect to the crack extension and tangent contact stresses acting on the facial surfaces of rigid inclusions. With the help of rotation operators, the resulting determining system of equations is reduced to a system of integral equations of the second kind of Fredholm type. The equation solution is constructed by the method of mechanical quadratures. A numerical analysis was carried out and regularities were revealed in the variation of the intensity factors of rupture stresses, crack extension and contact stresses under the inclusions depending on the physical and mechanical and geometrical characteristics of the problem

**Keywords**

*Mixed problem, disk-shaped crack, circular rigid inclusion*

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**Introduction.** Generating effective solutions of differential and integral equations of mathematical physics that simulate certain processes occurring in nature has always been and remains relevant from both theoretical and practical points of view. One of these equations is the Lame equations describing the stress-strain state of elastic bodies. General solutions of these equations for different areas are generated using different methods in both two-dimensional and three-dimensional cases. However, it is not often possible to obtain effective solutions to mixed boundary value problems using these solutions. In the axisymmetric case, when solving mixed boundary value problems, we have to use different integral operators, which makes it much more difficult to construct solutions and obtain useful information from a physical point of view. Many papers have been devoted to solving these problems of elasticity theory. A considerable part of the papers is devoted to the study of an axisymmetric stress state of a composite space consisting of two heterogeneous half-spaces with interfacial disc-shaped or ring-shaped defects. We should note the works [1–6] that are directly related to the paper research. Besides, a lot of the major results on axisymmetric contact and mixed problems of elasticity theory are given in [7, 8]. The study of axisymmetric mixed problems in the theory of elasticity for a piecewise homogeneous, uniformly layered space obtained by alternately joining two heterogeneous layers of the same thickness began relatively recently [9, 10]. In Ref. [10], discontinuous solutions of Lame differential equations are expressed for a uniformly layered space that contains a periodic system of circular disk-shaped parallel interfacial defects. Only [11] is devoted to the study of similar problems for a piecewise homogeneous, uniformly layered space with a periodic system of internal defects, which are also interesting and relevant, and examines the axisymmetric stress state of elastic, uniformly layered space with a periodic system of internal cracks.

**Problem statement and output of discontinuous solutions.** Consider the axisymmetric stress state of a uniformly layered elastic space consisting of two heterogeneous layers of $2h$ thickness with Lame coefficients $\lambda_1$, $\mu_1$ and $\lambda_2$, $\mu_2$. On the median planes of the first heterogeneous layer, expressed by the equations $z=(4n+1)h$ $(n \in Z)$, in a cylindrical coordinate system $Or\varphi z$ with a base plane $z=0$, aligned with one of the interfacing planes of heterogeneous layers, the space is relaxed by a system of periodic circular disk-shaped, parallel cracks of radii $a_1$. On the median planes of the second heterogeneous layer, expressed by the equations $z=(4n-1)h$ $(n \in Z)$, the space is strengthened by absolutely rigid disk-shaped , parallel inclusions of $a_2$ radii. We assume that the space is deformed under the influence of identical axisymmetric normal loads $P_1(r)$, acting on the banks of cracks.

It is necessary to determine the regularities of changes in crack extension and intensity coefficients of rupture stresses on the circles $r = a_1$, as well as contact stresses acting under inclusions depending on the physical-mechanical and geometric characteristics of heterogeneous layers.

Based on the problem statement all median planes of heterogeneous layers are planes of symmetry. Therefore, you can separate the basic cell as a two-component layer that occupies the space region $\Omega\{|z| \le h; 0 \le r < \infty; 0 \le \varphi \le 2\pi\}$ and formulate the task as a boundary problem for the basic cell. Figure 1 shows the axial section of the cell.

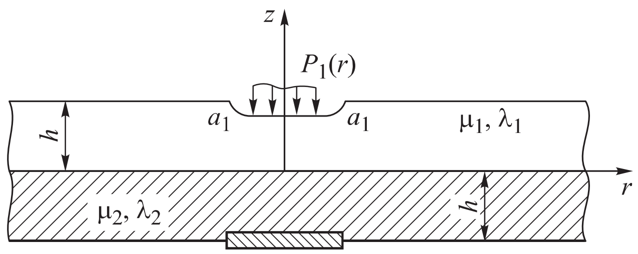


**Fig. 1.** Axial section of the basic cell

Let us put indexes 1 and 2 to all the variables, describing the stressed-strain state of the top, bottom layers of the cell, and represent the problem as a boundary value problem:

$$\begin{cases} u_r^{(1)}(r,0) = u_r^{(2)}(r,0) \quad (0 \le r < \infty); \\ u_z^{(1)}(r,0) = u_z^{(2)}(r,0) \quad (0 \le r < \infty); \\ \sigma_z^{(1)}(r,0) = \sigma_z^{(2)}(r,0) \quad (0 \le r < \infty); \\ \tau_{rz}^{(1)}(r,0) = \tau_{rz}^{(2)}(r,0) \quad (0 \le r < \infty); \end{cases} \tag{1a}$$

$$\begin{cases} \sigma_z^{(1)}(r,h) = -P_1(r); \quad \tau_{rz}^{(1)}(r,h) = 0 \quad (0 \le r < a_1); \\ u_z^{(2)}(r,-h) = 0; \quad u_r^{(2)}(r,-h) = 0 \quad (0 \le r < a_2); \\ \tau_{rz}^{(j)}\left(r, (-1)^{j+1} h\right) = 0; \quad u_z^{(j)}\left(r, (-1)^{j+1} h\right) = 0 \quad (a_j \le r < \infty), \quad j = 1, 2, \end{cases} \tag{1b}$$

where $u_r^{(j)}(r,z)$, $u_z^{(j)}(r,z)$ $(j = 1, 2)$ are radial and vertical displacements of layer points that satisfy Lame's differential equations; $\sigma_z^{(j)}(r,z)$, $\tau_{rz}^{(j)}(r,z)$

are normal and radial stresses acting in the corresponding layers and associated with displacements by known formulas [9].

Next, we present solutions of the Lame equations as Hankel integrals [11]:

$$u_r^{(j)}(r,z)=\int_0^\infty\left[\left(A_j(s)+zB_j^*(s)\right)\operatorname{ch}(zs)+\left(B_j(s)+zA_j^*(s)\right)\operatorname{sh}(zs)\right]sJ_1(rs)\,ds;$$
$$u_z^{(j)}(r,z)=\int_0^\infty\left[\left(C_j(s)-zA_j^*(s)\right)\operatorname{ch}(zs)+\left(D_j(s)-zB_j^*(s)\right)\operatorname{sh}(zs)\right]sJ_0(rs)\,ds. \quad (2)$$

Here

$$A_j^*(s)=\frac{s}{æ_j}\left(A_j(s)+D_j(s)\right);\;\; B_j^*(s)=\frac{s}{æ_j}\left(B_j(s)+C_j(s)\right)\;\;(j=1,2);$$

$J_j(x)$ $(j=0,1)$ is Bessel functions of a real argument; $A_j(s)$, $B_j(s)$, $C_j(s)$, $D_j(s)$ are unknown coefficients to be determined; $æ_j\,(j=1,2)$ are Muskhelishvili constants.

To solve the boundary problem (1), we introduce unknown functions for displacements of crack banks and tangent contact stresses acting under inclusion:

$$u_z(r,h)=-\frac{1}{2}w_1(r)\;\;(0<r<a_1);$$
$$\tau_{rz}^{(2)}(r,-h)=\tau_2(r)\;\;(0<r<a_2). \quad (3)$$

Let us solve an auxiliary boundary value problem consisting of conditions (1), in which the first and fourth conditions (1b) are replaced by conditions (3).

Using representations (2) and Hooke's law, we determine the components of stresses acting in heterogeneous layers. Then we satisfy the conditions of the auxiliary boundary value problem and determine the unknown coefficients $A_j(s)$, $B_j(s)$, $C_j(s)$, $D_j(s)$. We obtain

$$A_1^*(s)=-\frac{\left[\mu_2æ_1\vartheta_2^{(1)}-\mu_1(\mu_1-\mu_2)\beta\operatorname{th}\beta\right]}{2æ_1\vartheta_2^{(1)}\Delta_1(\beta)\operatorname{ch}\beta}s\,\overline{w}_1(s)-\frac{\overline{\tau}_2(s)}{2\Delta_1(\beta)\operatorname{ch}\beta};$$
$$A_2^*(s)=-\frac{\mu_1 s\,\overline{w}_1(s)}{2\Delta_2(\beta)\operatorname{ch}\beta}-\frac{\left[æ_2\vartheta_2^{(2)}+(\mu_1-\mu_2)\beta\operatorname{th}\beta\right]}{2æ_2\vartheta_2^{(2)}\Delta_2(\beta)\operatorname{ch}\beta}\overline{\tau}_2(s);$$
$$A_1(s)=A_2(s)=\frac{1}{\mu_1-\mu_2}\left[\frac{æ_1\vartheta_2^{(1)}}{s}A_1^*(s)-\frac{æ_2\vartheta_2^{(2)}}{s}A_2^*(s)\right];$$

$$C_1(s) = C_2(s) =$$

$$= -\frac{1}{\mu_1 - \mu_2}\left[-\operatorname{th}\beta\left(\frac{æ_1 \vartheta_2^{(1)}}{s} A_1^*(s) + \frac{æ_2 \vartheta_2^{(2)}}{s} A_2^*(s)\right) - \frac{\mu_1 \overline{w}_1(s)}{2\operatorname{ch}\beta} - \frac{\overline{\tau}_2(s)}{2\operatorname{sch}\beta}\right];$$

$$B_1^*(s) = -A_1^*(s)\operatorname{th}\beta - \frac{\mu_1 s \overline{w}_1(s)}{2æ_1 \vartheta_2^{(1)}\operatorname{ch}\beta}; \quad B_1^*(s) = A_2^*(s)\operatorname{th}\beta + \frac{\overline{\tau}_2(s)}{2æ_2 \vartheta_2^{(2)}\operatorname{ch}\beta};$$

$$B_j(s) = -C_j(s) + \frac{æ_j}{s} B_j^*(s); \quad D_j(s) = -A_j(s) + \frac{æ_j}{s} A_j^*(s).$$

Here

$$\Delta_j(\beta) = 2æ_j \vartheta_2^{(j)} \Delta_j^*(\beta); \quad \Delta_j^*(\beta) = \operatorname{th}\beta + (-1)^j (\mu_1 - \mu_2) E_j(\beta);$$

$$E_j(\beta) = \frac{1}{2\vartheta_2^{(j)}}\left[\operatorname{th}\beta - \frac{\beta}{æ_j \operatorname{ch}^2\beta}\right]; \quad \overline{w}_1(s) = \int_0^{a_1} r w_1(r) J_0(sr)\,dr;$$

$$\overline{\tau}_2(s) = \int_0^{a_2} r \tau_2(r) J_1(sr)\,dr;$$

$$\vartheta_1^{(j)} = \frac{\mu_j^2}{\lambda_j + 3\mu_j}; \quad \vartheta_2^{(j)} = \frac{\mu_j(\lambda_j + 2\mu_j)}{\lambda_j + 3\mu_j} \quad (\beta = hs; \ j = 1, 2).$$

Using the obtained coefficient values we determine the normal stresses acting on the banks of the cracks, and the radial displacements in the areas of contact of inclusions with the matrix through unknown crack extension functions $w_1(r)$ and tangential contact stresses acting on the facial surfaces of inclusions $\tau_2(x)$. We find

$$\begin{aligned} \sigma_z^{(1)}(r, h) &= -\vartheta_1 L_{0,0}^{(2)}[w_1] - L_{0,0}^{(2,1,1)}[w_1] + L_{0,1}^{(1,1,2)}[\tau_2]; \\ u_r^{(2)}(r, -h) &= -\frac{1}{\vartheta_2^{(2)}} L_{1,1}^{(0)}[\tau_2] + L_{1,0}^{(1,2,1)}[w_1] + L_{1,1}^{(0,2,2)}[\tau_2], \end{aligned} \tag{4}$$

where

$$L_{0,0}^{(2)}[w_1] = \int_0^{\infty} s^2 \overline{w}_1(s) J_0(sr)\,ds = \int_0^{a_1} W_{0,0}^{(2)}(r, \xi)\,\xi w_1(\xi)\,d\xi;$$

$$L_{1,1}^{(0)}[\tau_2] = \int_0^{\infty} \overline{\tau}_2(s) J_1(sr)\,ds = \int_0^{a_2} W_{1,1}^{(0)}(r, \xi)\,\xi \tau_2(\xi)\,d\xi;$$

$$W_{m,n}^{(k)}(r,\xi)=\int\limits_0^\infty t^k J_m(tr)J_n(t\xi)dt;$$

$$L_{m,n}^{(k,i,j)}[\varphi]=\int\limits_0^{a_j} W_{m,n}^{(k,i,j)}(r,\xi)\xi\varphi(\xi)d\xi;$$

$$W_{m,n}^{(k,i,j)}(r,\xi)=\int\limits_0^\infty K_{i,j}(th)t^k J_m(tr)J_n(t\xi)dt;$$

$$K_{1,1}(\beta)=\frac{\mu_1\operatorname{th}\beta\left[(\mu_1(æ_1+1)+æ_1\mu_2)\operatorname{sh}^2\beta+(æ_1+1)(\mu_1+\mu_2)/2\right]}{\mu_1(æ_1+1)\Delta_1^*(\beta)\Delta_2^*(\beta)\operatorname{ch}^2(\beta)}+$$

$$+\frac{Q_{1,1}(\beta)}{\mu_1(æ_1+1)\Delta_1^*(\beta)\Delta_2^*(\beta)\operatorname{ch}^2(\beta)}-\frac{\mu_1}{\vartheta_2^{(1)}}\left[\vartheta_1^{(1)}\operatorname{th}\beta+\frac{\mu_1\beta}{æ_1}\right]+$$

$$+\frac{2\beta\operatorname{th}\beta\left[(\mu_1+æ_1\mu_2)\operatorname{sh}^2\beta+\mu_2(æ_1+1)/2\right]}{(æ_1+1)^2\Delta_1^*(\beta)\operatorname{ch}^2(\beta)}-\frac{\mu_1}{2(1-\nu_1)};$$

$$K_{1,2}(\beta)=-\frac{\vartheta_2^{(1)}(\operatorname{th}\beta-\mu_1E_1-\mu_2E_1)+\mu_1\left(\operatorname{sh}^2\beta-\vartheta_2^{(1)}E_1\operatorname{sh}(2\beta)\right)}{2\vartheta_2^{(1)}\Delta_1^*(\beta)\Delta_2^*(\beta)\operatorname{ch}^2(\beta)}-$$

$$-\frac{\mu_1\beta\operatorname{th}\beta}{2æ_2\vartheta_2^{(2)}\Delta_2^*(\beta)\operatorname{ch}^2(\beta)};$$

$$K_{2,1}(\beta)=\frac{\operatorname{th}\beta-\mu_1E_1-\mu_2E_1}{4\Delta_1^*(\beta)\Delta_2^*(\beta)\operatorname{ch}^2(\beta)}+\frac{\mu_1\beta\operatorname{th}\beta}{2\operatorname{ch}^2(\beta)}\left(\frac{1}{\Delta_2(\beta)}+\frac{1}{\Delta_1(\beta)}\right);$$

$$K_{2,2}(\beta)=-\frac{E_1+E_2+2E_2\operatorname{sh}^2\beta}{4\Delta_1^*(\beta)\Delta_2^*(\beta)\operatorname{ch}^2(\beta)}+\frac{\operatorname{sh}^2\beta\left[æ_2+\beta\operatorname{th}\beta\left(1+(\mu_1-\mu_2)/\vartheta_2^{(2)}\right)\right]}{2\Delta_2(\beta)\operatorname{sh}^2\beta}+$$

$$+\frac{\beta\operatorname{cth}\beta}{2\Delta_2(\beta)}-\frac{\beta+æ_2\operatorname{th}\beta}{2æ_2\vartheta_2^{(2)}}+\frac{1}{2\vartheta_2^{(2)}};$$

$$Q_{1,1}(\beta)=-\mu_1\mu_2E_2\left[(\mu_1+æ_1\mu_2)\operatorname{sh}^2\beta+\frac{\mu_2(æ_1+1)}{2}\right]-\frac{\mu_1^3(æ_1+1)E_1}{2}+$$

$$+\mu_1^3(æ_1+1)\operatorname{sh}^2\beta\left[E_2-E_1-(\mu_1-\mu_2)E_1E_2\operatorname{cth}\beta\right]:\ \ \vartheta_1=\mu_1/2(1-\nu_1).$$

It is plain (enough), that the kernels $W_{m,n}^{(k,i,j)}(r,\xi)$ is the quadratically summable functions. Using the equations (4) we satisfy the first and fourth conditions (1b). As a result, to determine the displacements of the crack banks points and tangent contact stresses under inclusions, we write down a determining system of integral equations:

$$-\vartheta_1 L_{0,0}^{(2)}[w_1]-L_{0,0}^{(2,1,1)}[w_1]+L_{0,1}^{(1,1,2)}[\tau_2]=-P_1(r);$$
$$-\frac{1}{\vartheta_2^{(2)}}L_{1,1}^{(0)}[\tau_2]+L_{1,0}^{(1,2,1)}[w_1]+L_{1,1}^{(0,2,2)}[\tau_2]=0. \tag{5}$$

The equation system (5) is considered taking into account the conditions of continuity of displacements on the circles $r=a_1$:

$$w_1(a_1)=0. \tag{6}$$

**Solving the determining system of integral equations.** Let us start solving the system of integral equations (5). To do this [4–7] we introduce new required functions

$$w_*(t)=\frac{2}{\pi}\int_t^{a_1}\frac{\xi w_1(\xi)}{\sqrt{\xi^2-t^2}}d\xi; \qquad \tau_*(t)=\frac{2t}{\pi}\int_t^{a}\frac{\tau_2(\xi)}{\sqrt{\xi^2-t^2}}d\xi. \tag{7}$$

Let us continue them for an interval $(-a_j,0)$ $(j=1,2)$ by even and odd ways.

Then using the ratios

$$\overline{w}_1(s)=\int_0^{a_1} rw_1(r)J_0(sr)\,dr=\int_0^{a_1} w_*(t)\cos(st)\,dt;$$

$$\overline{\tau}_2(s)=\int_0^{a_2} r\tau_2(r)J_1(sr)\,dr=\int_0^{a_2}\tau_*(t)\sin(st)\,dt,$$

let us rewrite the (5) system of equations

$$\vartheta_1\int_0^{a_1}K(t,r)w_*'(t)\,dt+\int_0^{a_1}K_{1,1}(t,r)w_*'(t)\,dt+\int_0^{a_2}K_{1,2}(t,r)\tau_*(t)\,dt=-P_1(r);$$

$$-\frac{1}{\vartheta_2^{(2)}}\int_0^{a_1}Q(t,r)\tau_*(t)\,dt-\int_0^{a_1}K_{2,1}(t,r)w_*'(t)\,dt+\int_0^{a_2}K_{2,2}(t,r)\tau_*(t)\,dt=0. \tag{8}$$

Here

$$K(t,r)=\int_0^\infty sJ_0(sr)\sin(ts)\,ds;\; K_{1,i}(t,r)=\int_0^\infty K_{1,i}(hs)\,sJ_0(sr)\sin(ts)\,ds\quad (i=1,2);$$

$$Q(t,r)=\int_0^\infty J_1(sr)\sin(ts)\,ds;\; K_{2,i}(t,r)=\int_0^\infty K_{2,i}(hs)\,J_1(sr)\sin(ts)\,ds\quad (i=1,2).$$

Let us apply operators $I$ and $I_1$ to the first and second equations of (5) respectively:

$$I[\varphi(r)]=\int_0^x \frac{r\varphi(r)\,dr}{\sqrt{x^2-r^2}};\quad I_1[\varphi(x)]=\frac{d}{dx}\int_0^x \frac{y\,dy}{\sqrt{x^2-r^2}}\int_0^y \varphi(r)\,dr.$$

Then we differentiate the second of the obtained equations with respect to $x$, continuing both equations on the interval $(-a_j,0)$ in an odd way and taking into account the values of the known integrals [12, 13]:

$$\int_0^r J_1(rs)\,dr=-\frac{1}{s}[J_0(rs)-1];\quad \int_0^x \frac{J_0(rt)\,r\,dr}{\sqrt{x^2-r^2}}=\frac{\sin(xt)}{t};$$

$$\frac{d}{dx}\int_0^x \frac{J_0(rt)-1}{\sqrt{x^2-r^2}}\,r\,dr=\cos tx-1;\quad \int_0^\infty \sin(ts)\sin(sx)\,ds=\frac{\pi}{2}[\delta(t-x)-\delta(t+x)],$$

where $\delta(x)$ is known Dirac Delta function. We write the equation system (8) as

$$w'_*(x)+\int_{-a_1}^{a_1} R_{1,1}(t,x)\,w'_*(t)\,dt+\int_{-a_2}^{a_2} R_{1,2}(t,x)\,\tau_*(t)\,dt=f_1(x)\quad (-a_1<x<a_1);$$

$$\tau_*(x)+\int_{-a_1}^{a_1} R_{2,1}(t,x)\,w'_*(t)\,dt+\int_{-a_2}^{a_2} R_{2,2}(t,x)\,\tau_*(t)\,dt=0\quad (-a_2<x<a_2). \tag{9}$$

Therefore,

$$R_{1,i}(t,x)=\frac{1}{\pi\vartheta_1}\int_0^\infty K_{1,i}(hs)\sin(sx)\sin(st)\,ds;\quad f_1(x)=-\frac{2}{\pi\vartheta_1}I[P_1(r)];$$

$$R_{2,i}(t,x)=-\frac{4(-1)^i\vartheta_2^{(2)}}{\pi}\int_0^\infty K_{2,i}(hs)\sin(sx)\sin(st)\,ds\quad (i=1,2).$$

Thus, while using the formulas of rotation operators inversion [4–7] the function $w_1(x)$ can be written as

$$w_1(r) = -\frac{1}{r}\frac{d}{dr}\int_r^{a_1}\frac{sw_*(s)}{\sqrt{s^2-r^2}}ds, \tag{10}$$

and condition (6) is fulfilled automatically.

Using the change of variables $t = a_j\xi$, $x = a_j\eta$ $(j = 1, 2)$, we generate the system of equations (9) on an interval $(-1, 1)$, and determining $\varphi_1(\eta) = w'_*(a_1\eta)/a_1$, $\varphi_2(\eta) = \tau_*(a_2\eta)/\vartheta_1 a_2$ we obtain the system

$$\varphi_j(\eta) + \sum_{i=1}^{2}\int_{-1}^{1} Q^*_{j,i}(\xi,\eta)\varphi_i(\xi)d\xi = F_j(\eta) \quad (-1<\eta<1,\ j=1,2); \tag{11}$$

$$Q_{1,1}(\xi,\eta) = l_1 R_{11}(l_1\xi, l_1\eta) = \frac{l_1}{\pi\vartheta_1}\int_0^\infty K_{11}(\beta)\sin(l_1\beta\xi)\sin(l_1\beta\eta)d\beta;$$

$$Q_{1,2}(\xi,\eta) = \frac{\vartheta_1 l_2^2}{l_1}R_{12}(l_2\xi, l_1\eta) = \frac{l_2^2}{\pi l_1}\int_0^\infty K_{12}(\beta)\sin(l_2\beta\xi)\sin(l_1\beta\eta)d\beta;$$

$$Q_{2,1}(\xi,\eta) = \frac{l_1^2}{\vartheta_1 l_2}R_{21}(l_1\xi, l_2\eta) = \frac{4\vartheta_2^{(2)} l_1^2}{\pi\vartheta_1 l_2}\int_0^\infty K_{21}(\beta)\sin(l_1\beta\xi)\sin(l_2\beta\eta)d\beta;$$

$$Q_{2,2}(\xi,\eta) = l_2 R_{22}(l_2\xi, l_2\eta) = -\frac{4\vartheta_2^{(2)} l_2}{\pi}\int_0^\infty K_{21}(\beta)\sin(l_1\beta\xi)\sin(l_2\beta\eta)d\beta;$$

$$F_1(\eta) = f_1(a_1\eta)/a_1;\ F_2(\eta) = 0 \quad (l_j = a_j/h;\ \ j = 1,2).$$

Therefore, the solution of the problem was reduced to solving a system of integral equations of the second kind of Fredholm type (11). It is obvious that the solutions of this system of functions $\varphi_j(x)$ are restricted on circles $r = a_j$ $(j = 1, 2)$ respectively.

Let us write the formulas, which will help, after determining $\varphi_j(\eta)$ functions, to find crack extension and intensity coefficients of rupture stresses on the circles $r = a_1$.

To determine the dimensionless crack extension, we use the (10) equations and write

$$W(\eta) = \frac{w_1(a_1\eta)}{a_1} = -\int_\eta^1\frac{\varphi_1(\xi)}{\sqrt{\xi^2-\eta^2}}d\xi.$$

To determine the intensity coefficient of rupture stresses $K_I(a_1)$ on the circles $r = a_1$, let us represent the first equation of (4) while $(r > a_1)$ as

$$\sigma_z^{(1)}(r,h) = \vartheta_1 \int_0^{a_1} K(t,r) w_*'(t)\,dt + \int_0^{a_1} K_{1,1}(t,r) w_*'(t)\,dt + \int_0^{a_2} K_{1,2}(t,r) \tau_*(t)\,dt. \tag{12}$$

Using the equations [12]

$$sJ_0(rs) = \frac{1}{r}\frac{d}{dr}\left(rJ_1(rs)\right); \quad \int_0^\infty J_1(sr)\sin(ts)\,ds = \begin{cases} 0 & t > r; \\ \dfrac{t}{r}\dfrac{1}{\sqrt{r^2 - t^2}} & t < r, \end{cases}$$

we rewrite (12) as

$$\sigma_z^{(1)}(r,h) = \frac{\vartheta_1}{r}\frac{d}{dr}\int_0^{a_1} \frac{t w_*'(t)\,dt}{\sqrt{r^2 - t^2}} + \int_0^{a_1} K_{1,1}(t,r) w_*'(t)\,dt + \int_0^{a_2} K_{1,2}(t,r)\tau_*(t)\,dt =$$

$$= -\vartheta_j w_*'(a_j) / \sqrt{r^2 - a_j^2} + F_1(r). \tag{13}$$

Here

$$F_1(r) = \int_0^{a_1} K_{1,1}(t,r) w_*'(t)\,dt + \int_0^{a_2} K_{1,2}(t,r)\tau_*(t)\,dt +$$

$$+ \frac{\vartheta_1}{r}\frac{d}{dr}\int_0^{a_j} \frac{t\left[w_*'(t) - w_*'(a_1)\right]dt}{\sqrt{r^2 - t^2}} + \frac{\vartheta_1 w_*'(a_1)}{r}$$

is restricted functions on a circle $r = a_1$. Using (13), we find

$$K_I(a_1) = \lim_{r \to a_1 + 0} \sqrt{2(r - a_1)}\,\sigma_z^{(1)}(r,h) = -\frac{\vartheta_1 w_1'(a_1)}{\sqrt{a_1}} = -\sqrt{a_1}\,\vartheta_1\,\varphi_1(1).$$

Therefore, we determine the dimensionless coefficients of the rupture stresses intensities on the circles $r = a_1$ using the formula:

$$K_I^*(a_1) = \frac{K_I(a_1)}{\sqrt{a_1}\mu_1} = -\frac{\varphi_1(1)}{2(1 - \nu_1)}. \tag{14}$$

Let us express the formula for determining normal and tangent contact stresses acting under inclusions. First, we find the tangent contact stresses. To do this, we use the inversions of the second integral operator from (7) and define

$$\tau_2(r) = -\frac{d}{dr}\int_r^a \frac{\tau_*(s)}{\sqrt{s^2 - r^2}}\, ds.$$

Hence for the dimensionless contact stresses $\tau(\eta) = \tau_2(a_2\eta)/\vartheta_1$ we get

$$\tau(\eta) = -\frac{d}{d\eta}\int_\eta^1 \frac{\varphi_2(\xi)}{\sqrt{\xi^2 - \eta^2}}\, d\xi.$$

To determine normal contact stresses, we calculate $\sigma_z^{(2)}(r, -h)$. Then,

$$\sigma_z^{(2)}(r,-h) = \frac{\vartheta_1^{(2)}}{\vartheta_2^{(2)} r}\frac{d}{dr}\int_0^r \frac{t\tau_*(t)\,dt}{\sqrt{r^2 - t^2}} + \int_0^{a_1} K_{3,1}(t,r)\, w_*'(t)\, dt +$$

$$+ \int_0^{a_2} K_{3,2}(t,r)\,\tau_*(t)\, dt, \qquad (15)$$

$$K_{3,i}(t,r) = \int_0^\infty K_{3,i}(hs)\, sJ_0(sr)\sin(ts)\, ds \quad (i = 1,2);$$

$$K_{3,1}(\beta) = \frac{\mu_2 \operatorname{th}\beta\left[æ_2\mu_1 \operatorname{sh}^2\beta + (æ_2+1)(\mu_1+\mu_2)/2\right] + Q_{3,1}(\beta)}{\mu_2(æ_2+1)\Delta_1^*(\beta)\Delta_2^*(\beta)\operatorname{ch}^2(\beta)} +$$

$$+ \frac{\mu_2\beta \operatorname{th}\beta}{(æ_1+1)\Delta_1^*(\beta)\operatorname{ch}^2(\beta)};$$

$$Q_{3,1}(\beta) = -\mu_1\mu_2 E_1\left[(\mu_2 + æ_2\mu_1)\operatorname{sh}^2\beta + \frac{\mu_1(æ_2+1)}{2}\right] - \frac{\mu_2^3(æ_2+1)E_2}{2} -$$

$$-\mu_1\mu_2^2(æ_2+1)\operatorname{sh}^2\beta\left[E_2 - E_1 - (\mu_1-\mu_2)E_1E_2 \operatorname{cth}\beta\right];$$

$$K_{3,2}(\beta) = -\frac{\vartheta_2^{(2)}(\operatorname{th}\beta - \mu_1E_1 - \mu_2E_2) + \mu_2\Delta_1^*(\beta)\left[\operatorname{sh}^2\beta - \vartheta_2^{(2)}E_2\operatorname{sh}^2\beta\right]}{2\vartheta_2^{(2)}\Delta_1^*(\beta)\Delta_2^*(\beta)\operatorname{ch}^2(\beta)} +$$

$$+\frac{\vartheta_1^{(2)}(\operatorname{th}\beta - 1)}{\vartheta_2^{(2)}} - \frac{\beta\operatorname{th}\beta\left[\mu_2(\mu_2 + æ_2\mu_1)\operatorname{sh}^2\beta + æ_2\mu_1\vartheta_2^{(2)}\right]}{æ_2\vartheta_2^{(2)}\Delta_2(\beta)\operatorname{ch}^2(\beta)} + \frac{\mu_2\beta}{æ_2\vartheta_2^{(2)}}.$$

Writing (15) with the help of functions $\varphi_j(\xi)$ $(j = 1,2)$, we obtain the following expression for the dimensionless contact stresses:

$$\sigma(\eta)=\frac{\sigma_z^{(2)}(r,-h)}{\vartheta_1}=\frac{\vartheta_1^{(2)}}{\vartheta_2^{(2)}\eta}\frac{d}{d\eta}\int_0^\eta\frac{\xi\varphi_2(\xi)d\xi}{\sqrt{\eta^2-\xi^2}}+$$

$$+\int_0^1 R_{3,1}(\xi,\eta)\varphi_1(\xi)d\xi+\int_0^1 R_{3,2}(\xi,\eta)\varphi_2(\xi)d\xi, \qquad (16)$$

$$R_{3,1}(\xi,\eta)=\frac{l_1^2}{\vartheta_1}\int_0^\infty K_{3,1}(\beta)\beta J_0(l_2\beta\eta)\sin(l_1\beta\eta)d\beta;$$

$$R_{3,2}(\xi,\eta)=l_1^2\int_0^\infty K_{3,2}(\beta)\beta J_0(l_2\beta\eta)\sin(l_2\beta\eta)d\beta.$$

Let us note, that using the ratios [12, 13]

$$sJ_0(sr)=\frac{1}{r}\frac{d}{dr}\left[rJ_1(sr)\right];\quad J_1(s\xi)=\frac{2}{\pi\xi}\int_0^\xi\frac{t\sin ts}{\sqrt{\xi^2-t^2}}dt,$$

we can write (16) as

$$\sigma(\eta)=\frac{2}{\pi\eta}\frac{d}{d\eta}\int_0^\eta\frac{\xi}{\sqrt{\eta^2-\xi^2}}\times$$

$$\times\left\{\frac{\pi\vartheta_1^{(2)}}{2\vartheta_2^{(2)}}\varphi_2(\eta)+\int_0^1 R_{3,1}^*(\xi,\eta)\varphi_1(\xi)d\xi+\int_0^1 R_{3,2}^*(\xi,\eta)\varphi_2(\xi)d\xi\right\}d\xi,$$

$$R_{3,1}^*(\xi,\eta)=\frac{l_1^2}{\vartheta_1 l_2}\int_0^\infty K_{3,1}(\beta)\sin(l_1\beta\xi)\sin(l_2\beta\eta)d\xi;$$

$$R_{3,2}^*(\xi,\eta)=l_2\int_0^\infty K_{3,2}(\beta)\sin(l_2\beta\xi)\sin(l_2\beta\eta)d\xi.$$

**Numerical calculation.** Using the method of mechanical quadratures [14, 15], we carried out a numerical calculation and studied the regularities of changes in the dimensionless coefficient of rupture stresses intensity $K_I^*(a_1)$ on a circle $r=a_1$, crack extension $W(x)$, dimensionless normal and tangent stresses, active on the facial sides of inclusions $\sigma(\eta)$ and $\tau(\eta)$ depending on the parameter change $l=h/a$ in case, when $\mu=\mu_2/\mu_1=0.5$, $a_1=a_2=a$, $\nu_1=0.25$, $\nu_2=0.3$, $P_1(r)=P_0=\text{const}$, $P_1^*=P_0/\mu_1=0.1$. In this case $F_1(\eta)=-4(1-\nu_1)P_1^*\eta/\pi$.

Table 1 shows values of the dimensionless intensity coefficient $K_I^*(a_1)$ depending on the parameter $l=h/a$. While increasing $l$, which can be interpreted as increasing the layers height $h$ at a constant $a$, the intensity coefficient first increases and then decreases, tending to a certain limit value. The limit value corresponds to the homogeneous space made of the first material with a single disc-shaped crack.

*Table 1*

**Values of the dimensionless intensity coefficient $K_I^*(a_1)$ depending on the parameter $l = h / a$**

| $l$ | 0.1 | 0.5 | 1 | 2 | 10 | 100 |
|---|---|---|---|---|---|---|
| $K_I^*(a)$ | 0.02777 | 0.04728 | 0.07106 | 0.06779 | 0.06668 | 0.06666 |

Figure 2 shows the plots of crack extension $W(\eta)$ and normal contact stresses $\sigma(\eta)$ versus $l$. While increasing $l$, i.e., when removing cracks from inclusions, the crack extension first increases and then decreases, tending to a certain limit value. This value corresponds to the extension of a single disc-shaped crack in a homogeneous space made of the first layer material. At the same time, the normal contact stresses decrease in absolute value, tending to zero.

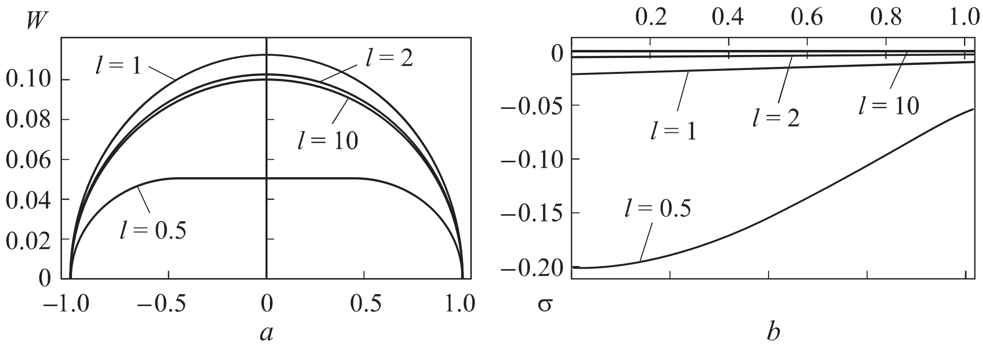


**Fig. 2.** Crack opening $W(\eta)$ (*a*) and normal contact stresses $\sigma(\eta)$ (*b*) versus $l$

Figure 3 shows the plots of crack extension and normal contact stresses versus parameter $\mu$ at $l = 0.5$, $a_1 = a_2 = a$, $\nu_1 = 0.4$, $\nu_2 = 0.25$, $P_1^* = 0.1$. In this case, when increasing $\mu$ which means increase of $\mu_2$ at a constant $\mu_1$ the crack extension reduces. Normal contact stresses increase in absolute value.

Table 2 shows values of the dimensionless intensity coefficient $K_I^*(a)$ depending on $\lambda$ parameter. Figure 4 shows the plots of normal and tangential stresses versus parameter $\lambda = a_2 / a_1$ at $a_1 / h = 1$, $\mu = 0.5$, $\nu_1 = 0.25$, $\nu_2 = 0.3$, $P_1^* = 0.1$. In this case, with increase in $\lambda$, which can be interpreted as increase in $a_2$ at constant $a_1$, the intensity coefficients of rupture stresses fluctuate

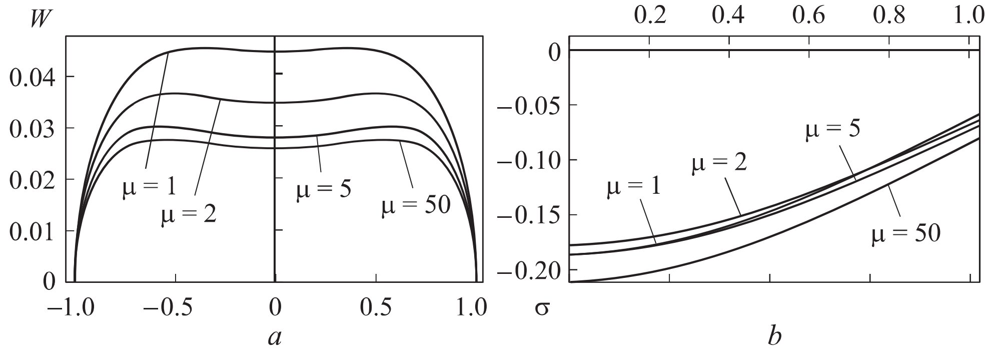


**Fig. 3.** Crack opening $W(\eta)$ (*a*) and normal contact stresses $\sigma(\eta)$ (*b*) versus $\mu$

around the values obtained at $\lambda = 1$. Contact stresses become alternating, i.e., tensile normal stresses appear in the contact areas of inclusions with the matrix, which can lead to separation of inclusions from the matrix.

*Table 2*

**Values of the dimensionless intensity coefficient $K_I^*(a)$ depending on the λ parameter**

| $\lambda$ | 0.1 | 1 | 4 | 8 | 15 |
|---|---|---|---|---|---|
| $K_I^*(a)$ | 0.06946 | 0.07106 | 0.07628 | 0.06619 | 0.07923 |

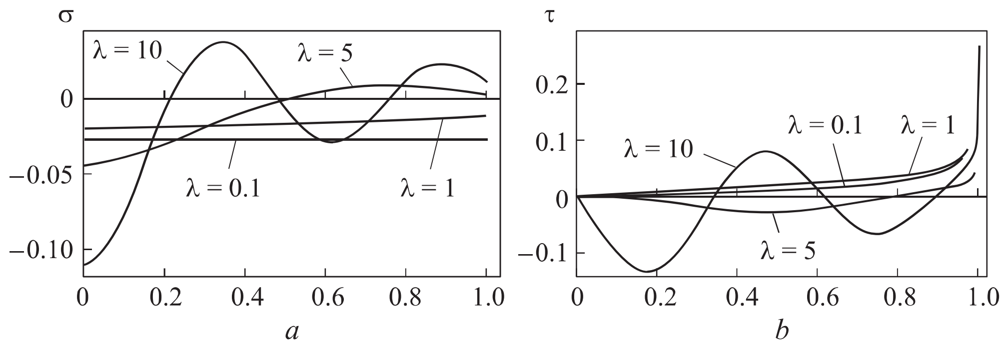


**Fig. 4.** Normal (*a*) and tangential (*b*) contact stresses versus $\lambda = a_2 / a_1$

**Conclusion.** The article provides a solution to the problem of axisymmetric stress state of a uniformly layered composite with periodic systems of circular disk-shaped parallel internal cracks and inclusions. Simple formulas are obtained for determining important mechanical characteristics of the problem, such as facial surface stresses of inclusions, crack extension, and the intensity coefficient of rupture stresses. Using numerical analysis, we showed that the interaction of cracks and inclusions increases when they approach each other. In the case of removing cracks from inclusions, the cracks work as separate, single disc-shaped cracks in a homogeneous space. Facial surface stresses of the inclusions disappear. It was found that increasing stiffness of the second layer, when the first layer stiffness is constant, leads to a decrease in both the intensity coefficients and the crack extension. The results of computational work revealed that when the radius of inclusions increases, there are zones in the contact areas between inclusions and the matrix, where normal surface stresses become tensile, which can lead to separation of inclusions from the matrix.

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