SOLVING TERMINAL PROBLEMS FOR MULTIDIMENSIONAL AFFINE SYSTEMS BASED ON TRANSFORMATION TO A QUASICANONICAL FORM

D.A. Fetisov
Bauman Moscow State Technical University, Moscow, Russian Federation
e-mail: dfetisov@yandex.ru

The paper considers a terminal problem for multidimensional affine systems, which are not linearizable by a feedback. The affine system is transformed to a regular quasicanonical form using a smooth nondegenerate change of variables within the range of states. In addition, the terminal problem for the initial system is transformed to the equivalent terminal problem for the system of a quasicanonical form. A method of solving the terminal problems is proposed for the quasicanonical systems, which is based on a concept of dynamics inverse problems generalization. The sufficient condition for applying the proposed method is proved. The numerical procedure of solving the terminal problems for the systems of a quasicanonical form is proposed. There is an example of solution development of a terminal problem for a sixth-order system using the above-mentioned method. The obtained results may be used for solving problems of terminal control over technical systems.

***Keywords*:** affine system, control, quasicanonical form, terminal problem.

Introduction. Equivalent transformations of the systems with a control provide many opportunities for solving various control theory problems. Papers [1–3] present methods of controllability research, construction of reachability sets, solving the problems of stabilization, and the terminal problems based on the system transformation to certain canonical forms. In this paper, the issue of terminal problems solution to the affine systems is considered. Different approaches to this issue can be found in [1, 4, 5–9]. Papers [1, 4] describe methods of terminal problems solutions of the affine systems, which are linearizable by a feedback, i.e. the systems that are converted to linear controlled systems by a smooth nondegenerate change of variables and a reversible change of controls. The methods for solving the terminal problems of the linear controlled systems are well known and are based on the application of the concept of dynamics inverse problems [10]. Nowadays, one of the most important challenges is the development of methods for solving the terminal problems of the systems, which are not linearizable by a feedback. Papers [5–8] set out the methods for solving the terminal problems for such systems. However, these methods cover a relatively small class of systems; the range of applicability of such methods imposes severe restrictions on system dimensions. A special kind of the system vector fields is often used. Thus, the issue of solving the terminal problems for the affine systems, which are not linearizable by a feedback is relevant. The present paper is dedicated to this problem.

Let us consider the following problem. For an affine system

$$\dot{x} = F(x) + \sum_{j=1}^{m} G_j(x)u_j; \tag{1}$$

$$x = (x_1, \ldots, x_n)^{\mathrm{T}} \in R^n, \quad u = (u_1, \ldots, u_m)^{\mathrm{T}} \in R^m;$$
$$F(x) = (F_1(x), \ldots, F_n(x))^{\mathrm{T}}, \quad G_j(x) = (G_{1j}(x), \ldots, G_{nj}(x))^{\mathrm{T}};$$
$$F_i(x),\ G_{ij}(x) \in C^{\infty}(R^n), \quad i = \overline{1,n}, j = \overline{1,m},$$

which is not linearizable by a feedback, it is required to find such continuous controls $u_1 = u_1(t), \ldots, u_m = u_m(t)$, $t \in [0, t_*]$ that for given time t_* can transform system (1) from the initial state $x(0) = x_0$ to the final state $x(t_*) = x_*$.

Transformation of the system to a quasicanonical form. The following theorem [11] sets the necessary and sufficient conditions under which system (1) is transformed to a quasicanonical form

$$\begin{aligned}
&\dot{z}_1^i = z_2^i;\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
&\dot{z}_{r_i-1}^i = z_{r_i}^i;\\
&\dot{z}_{r_i}^i = f_i(z^1, \ldots, z^m, \eta) + \sum_{j=1}^{m} g_{ij}(z^1, \ldots, z^m, \eta) u_j, \quad i = \overline{1,m};\\
&\dot{\eta} = q(z^1, \ldots, z^m, \eta);
\end{aligned} \qquad (2)$$

$$r_1 + \ldots + r_m = n - \rho, \quad z^i = (z_1^i, \ldots, z_{r_i}^i)^{\mathrm{T}}, \quad \eta = (\eta_1, \ldots, \eta_\rho)^{\mathrm{T}};$$
$$q(z^1, \ldots, z^m, \eta) = (q_1(z^1, \ldots, z^m, \eta), \ldots, q_\rho(z^1, \ldots, z^m, \eta))^{\mathrm{T}}.$$

In the formulation of the theorem, vector fields are used

$$F = \sum_{i=1}^{n} F_i(x) \frac{\partial}{\partial x_i}, \quad G_j = \sum_{i=1}^{n} G_{ji}(x) \frac{\partial}{\partial x_i}, \quad j = \overline{1,m},$$

which one-to-one correspond to system (1) in the range of states R^n and the vector fields $\mathrm{ad}_F^0 G_j = G_j$, $\mathrm{ad}_F^k G_j = [F, \mathrm{ad}_F^{k-1} G_j]$, $k = 1, 2, \ldots$, where $[X, Y]$ is a commutator of the vector fields X and Y.

Theorem 1. *For the transformation of the affine system* (1) *on the set* $\Omega \subseteq R^n$ *to a quasicanonical form* (2) *it is necessary and sufficient to have the following features*:

1) *functions* $\varphi_i(x) \in C^{\infty}(\Omega)$, $i = \overline{1,m}$, *satisfying the system of the first-order partial differential equations in the set* Ω

$$\mathrm{ad}_F^k G_j \varphi_i(x) = 0, \quad k = \overline{0, r_i - 2}, i, j = \overline{1,m}, \quad x \in \Omega;$$

2) *functions* $\varphi_{n-\rho+l}(x) \in C^{\infty}(\Omega)$, $l = \overline{1,\rho}$ *that for all* $x \in \Omega$

$$G_j \varphi_{n-\rho+l}(x) = 0, \quad j = \overline{1,m}, \quad l = \overline{1,\rho}$$

and mapping $\Phi : \Omega \to \Phi(\Omega)$, *prescribed by the system of functions*

$$z_k^i = F^{k-1}\varphi_i(x), \quad k = \overline{1, r_i}, \quad i = \overline{1, m};$$
$$\eta_l = \varphi_{n-\rho+l}(x), \quad l = \overline{1, \rho},$$

was a diffeomorphism.

In variables z^1, ..., z^m, η, system (1) has a quasicanonical form (2). If the matrix of the coefficients in system controls (2)

$$g(z^1, \ldots, z^m, \eta) = \begin{pmatrix} g_{11}(z^1, \ldots, z^m, \eta) & \ldots & g_{1m}(z^1, \ldots, z^m, \eta) \\ \vdots & \ddots & \vdots \\ g_{m1}(z^1, \ldots, z^m, \eta) & \ldots & g_{mm}(z^1, \ldots, z^m, \eta) \end{pmatrix}$$

is nondegenerated on the set $\Phi(\Omega)$, then system (2) is called regular on the set $\Phi(\Omega)$.

We will assume that system (1) satisfies the conditions of theorem 1, while $\Phi(\Omega) = R^n$. Then system (1) will be transformed to an equivalent of a quasicanonical form (2), which is determined on the whole range of states, and the terminal problem for system (1) — to the equivalent terminal problem for system (2): to find continuous controls $u_1 = u_1(t), \ldots, u_m = u_m(t)$, $t \in [0, t_*]$, transforming system (2) for time t_* from the initial state

$$\Phi(x_0) = (z_0^1, \ldots, z_0^m, \eta_0) \tag{3}$$

to the final state

$$\Phi(x_*) = (z_*^1, \ldots, z_*^m, \eta_*). \tag{4}$$

Controls $u_1 = u_1(t), \ldots, u_m = u_m(t)$, which are the solution to problem (3), (4) for system (2), simultaneously constitute the solution to the initial terminal problem for system (1). In this connection, we will consider terminal problem (3), (4) for system (2).

The solution to the terminal problem for the system of a quasicanonical form. Paper [9] describes the following necessary and sufficient condition for the existence of the terminal problem solution for a regular system of a quasicanonical form.

Theorem 2. *For the continuous controls* $u_1 = u_1(t), \ldots, u_m = u_m(t)$, $t \in [0, t_*]$ *to exist, which are the solution of terminal problem* (3), (4) *for regular system* (2), *it is necessary and sufficient for the functions* $B_i(t) \in C^{r_i}([0, t_*])$, $i = \overline{1, m}$ *to exist, and*:

1) *vector-functions* $\overline{B}_i(t) = \left(B_i(t), B_i'(t), \ldots, B_i^{(r_i-1)}(t)\right)^{\mathrm{T}}$ *satisfies the conditions*

$$\overline{B}_i(0) = z_0^i, \quad \overline{B}_i(t_*) = z_*^i;$$

2) *the solution* $\eta(t)$ *of the Cauchy problem*

$$\dot{\eta} = q(\overline{B}_1(t), \ldots, \overline{B}_m(t), \eta), \quad \eta(0) = \eta_0 \tag{5}$$

is determined at all $t \in [0, t_*]$ *and satisfies the condition*

$$\eta(t_*) = \eta_*. \tag{6}$$

In paper [9] it is also shown that the control $u = u(t)$, which is the solution to the terminal problem, is found according to the equality

$$u(t) = g^{-1}(\overline{B}_1(t), \ldots, \overline{B}_m(t), \eta(t)) \times \\ \times \begin{pmatrix} B_1^{(r_1)}(t) - f_1(\overline{B}_1(t), \ldots, \overline{B}_m(t), \eta(t)) \\ \ldots \\ B_m^{(r_m)}(t) - f_m(\overline{B}_1(t), \ldots, \overline{B}_m(t), \eta(t)) \end{pmatrix}, \tag{7}$$

while the relations $z^i = \overline{B}_i(t)$, $i = \overline{1, m}$, $\eta = \eta(t)$, $t \in [0, t_*]$ are the parametric equations of that phase trajectory of system (2) which connects states (3) and (4).

According to [9], we shall find functions $B_1(t), \ldots, B_m(t)$ from theorem 2 in the form of

$$B_i(t) = b_i(t) + c_i d_i(t), \qquad i = \overline{1, m},$$

where $b_i(t)$, $d_i(t) \in C^{r_i}([0, t_*])$, the vector-functions

$$\overline{b}_i(t) = \left(b_i(t), b_i'(t), \ldots, b_i^{(r_i-1)}(t)\right)^{\mathrm{T}}$$

satisfy the conditions

$$\overline{b}_i(0) = z_0^i, \quad \overline{b}_i(t_*) = z_*^i, \quad i = \overline{1, m},$$

while the vector-functions $\overline{d}_i(t) = \left(d_i(t), d_i'(t), \ldots, d_i^{(r_i-1)}(t)\right)^{\mathrm{T}}$ satisfy the conditions

$$\overline{d}_i(0) = 0, \quad \overline{d}_i(t_*) = 0, \quad i = \overline{1, m}, \tag{8}$$

It is necessary to find $c_i \in R$.

It is possible, for example, to take interpolation polynomials of $2r_i - 1$ degrees as functions $b_i(t)$, $i = \overline{1, m}$, and to take any functions, for which correlations (8) are fulfilled as functions $d_i(t)$, $i = \overline{1, m}$. With the given set of functions $B_i(t)$, condition 1 of theorem 2 is fulfilled for any $c_i \in R$. Numbers c_i should be selected in such a way that condition 2 of theorem 2 was fulfilled. If there exist such numbers as $c_1 = c_{1*}, \ldots, c_m = c_{m*}$ that the solution $\eta(t)$ of the Cauchy problem (5) satisfies the additional requirement

$\eta(t_*) = \eta_*$, then, for functions $B_i(t) = b_i(t) + c_{i*}d_i(t)$, $i = \overline{1, m}$, all conditions of theorem 2 are fulfilled and, thus, terminal problem (3), (4) for system (2) has got a solution.

Let us assume that $\rho \leqslant m$. The Euclidean norm will be considered as the vector norm from the space R^ρ and $\rho\times\rho$-matrices. Let $r = \max\{r_1, \ldots, r_\rho\}$. For all pairs of the indices l и j, where $l \in \{2, \ldots, r\}$, $j \in \{1, \ldots, \rho\}$, $l > r_j$, we will introduce the formally additional variables z_l^j. Let us denote $z_l = (z_l^1, \ldots, z_l^\rho)^{\mathrm{T}}$, $l = \overline{1, r}$. According to the definition, let us assume that if $l > r_j$, then $\partial q_i/\partial z_l^j = 0$ for all $i = \overline{1, \rho}$. Let us take $\partial q/\partial z_l$ for denoting $\rho \times \rho$-matrices with elements $\partial q_i/\partial z_l^j$, $i, j = \overline{1, \rho}$.

Irrespective of the number i, we specify the functions $d_i(t)$ with the formula

$$d_i(t) \equiv d(t) = \frac{t^r(t_* - t)^r}{\int\limits_0^{t_*} t^r(t_* - t)^r dt}. \tag{9}$$

We will denote $L = \max\limits_{[0,t_*]}\{d(t) + |d'(t)| + |d''(t)| + \ldots + |d^{(r-1)}(t)|\}$.

Let us prove the following auxiliary statement.

Lemma 1. *Let $P(t)$, $R(t)$ be $\rho \times \rho$-matrices with elements $P_{ij}(t)$, $R_{ij}(t) \in C[0, t_*]$, and there exists such a number $\lambda \in R$, with all $y \in R^\rho$, $t \in [0, t_*]$, the inequality is fulfilled*:

$$(P(t)y, y) \leqslant \lambda\|y\|^2. \tag{10}$$

Then $\rho \times \rho$-matrix $W(t)$, which is the solution to the Cauchy problem

$$\dot W = P(t)W + R(t), \qquad W(0) = 0, \tag{11}$$

satisfies the inequality

$$\|W(t_*)\| \leqslant e^{\lambda t_*}\int\limits_0^{t_*}\|R(t)\|e^{-\lambda t}dt. \tag{12}$$

◀ We denote j-th matrix columns $W(t)$ and $R(t)$ with $W_j(t)$ and $R_j(t)$, respectively. Then the system $\dot W = P(t)W + R(t)$ can be written in the following form:

$$\begin{pmatrix} \dot W_1 \\ \dot W_2 \\ \ldots \\ \dot W_\rho \end{pmatrix} = \begin{pmatrix} P(t) & 0 & \ldots & 0 \\ 0 & P(t) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & P(t) \end{pmatrix} \begin{pmatrix} W_1 \\ W_2 \\ \ldots \\ W_\rho \end{pmatrix} + \begin{pmatrix} R_1(t) \\ R_2(t) \\ \ldots \\ R_\rho(t) \end{pmatrix}.$$

Let us denote

$$V(t)=\begin{pmatrix} W_1(t)\\ W_2(t)\\ \dots\\ W_\rho(t)\end{pmatrix},\quad Q(t)=\begin{pmatrix} P(t) & 0 & \dots & 0\\ 0 & P(t) & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & P(t)\end{pmatrix},\quad S(t)=\begin{pmatrix} R_1(t)\\ R_2(t)\\ \dots\\ R_\rho(t)\end{pmatrix}$$

and put down Cauchy problem (11) in the form of

$$\dot{V}=Q(t)V+S(t),\qquad V(0)=0.$$

As the Euclidean norms of the matrixes $W(t)$ and $R(t)$ coincide with the Euclidean norms of the vectors $V(t)$ and $S(t)$, then, to prove inequality (12) it is sufficient to show that

$$\|V(t_*)\|\leqslant e^{\lambda t_*}\int\limits_0^{t_*}\|S(t)\|e^{-\lambda t}dt. \tag{13}$$

As it follows from inequality (10), for any $t\in[0,t_*]$ and $V= =(V_1^{\mathrm{T}},\dots,V_\rho^{\mathrm{T}})^{\mathrm{T}}\in R^{\rho^2}$, where $V_j\in R^\rho$, the estimation

$$\begin{gathered}(Q(t)V,V)=(P(t)V_1,V_1)+\dots+(P(t)V_\rho,V_\rho)\leqslant\\ \leqslant\lambda\|V_1\|^2+\dots+\lambda\|V_\rho\|^2=\lambda\|V\|^2.\end{gathered} \tag{14}$$

is true.

Let us use (14) to prove inequality (13). Note that, if $V(t_*)=0$, then $\|V(t_*)\|=0$ and the correctness of inequality (13) results from the non-negativity of its right part.

If $V(t_*)\neq 0$, then we can denote as t_0 the exact upper boundary of these t from the range $[0;t_*)$, for which $V(t)=0$. Then $V(t_0)=0$, for all $t\in(t_0;t_*)$ the inequality $V(t)\neq 0$ is fulfilled. In the range $(t_0;t_*)$ we can calculate and estimate $\frac{d}{dt}\|V\|$, using inequality (14) and the Cauchy–Bunyakowsky inequality:

$$\frac{d}{dt}\|V\|=\frac{(V,\dot{V})}{\|V\|}=\frac{1}{\|V\|}\left[(Q(t)V,V)+(S(t),V)\right]\leqslant$$

$$\leqslant\frac{\lambda\|V\|^2}{\|V\|}+\left(S(t),\frac{V}{\|V\|}\right)\leqslant\lambda\|V\|+\|S(t)\|.$$

Thus, in the range $(t_0;t_*)$ the function $\|V(t)\|$ satisfies the differential inequality

$$\frac{d}{dt}\|V\|\leqslant\lambda\|V\|+\|S(t)\|.$$

The following function is the solution to the differential equation $\dot{v} = \lambda v + \|S(t)\|$ with the initial condition $v(t_0) = 0$:

$$v(t) = e^{\lambda t} \int_{t_0}^{t} \|S(\tau)\| e^{-\lambda\tau} d\tau,$$

That is why, with all $t \in [t_0, t_*]$ inequality (12) is true:

$$\|V(t)\| \leqslant e^{\lambda t} \int_{t_0}^{t} \|S(\tau)\| e^{-\lambda\tau} d\tau$$

and, consequently,

$$\|V(t_*)\| \leqslant e^{\lambda t_*} \int_{t_0}^{t_*} \|S(t)\| e^{-\lambda t} dt. \tag{15}$$

From the non-negativity of the subintegral function in the right part of inequality (15) we have

$$\int_{t_0}^{t_*} \|S(t)\| e^{-\lambda t} dt \leqslant \int_{0}^{t_*} \|S(t)\| e^{-\lambda t} dt,$$

and with this, we get inequality (13) from (15).

Now we can prove the main result.

Theorem 3. Let us assume the following:

1) $q(z^1, \ldots, z^m, \eta) = \sum\limits_{i=1}^{r} A_i z_i + K\eta + p(z^1, \ldots, z^m)$ *where* $A_1, \ldots, A_r$, K *are* $\rho \times \rho$*-matrixes*;

2) *matrix* $M = A_1 + KA_2 + K^2 A_3 + \ldots + K^{r-1} A_r$ *is nondegenerated*;

3) *there is such* $\varepsilon > 0$, *that for all* $i = \overline{1, r}$ *and* $(z^1, \ldots, z^m) \in R^{n-\rho}$ *the inequalities* $\|\partial p / \partial z_i\| \leqslant \varepsilon$ *are fulfilled*;

4) λ *is the largest proper number of the matrix* $(P + P^{\mathrm{T}})/2$, *where* $P = M^{-1} K M$;

$$\gamma = \begin{cases} (\|M^{-1}\| \varepsilon L + \|P\|) t_*, & \text{if } \lambda = 0; \\ (\|M^{-1}\| \varepsilon L + \|P\|) \dfrac{e^{\lambda t_*} - 1}{\lambda}, & \text{if } \lambda \neq 0. \end{cases} \tag{16}$$

If $\gamma < 1$, *then terminal problem (3), (4) for system (2) has got a solution.*

◀ Let us assume $c_{\rho+1} = \ldots = c_m = 0$, then denote the vector with unknown parameters by $c = (c_1, \ldots, c_\rho)^{\mathrm{T}}$. Then Cauchy problem (5) will take the form of

$$\begin{aligned} &\dot{\eta} = q(\bar{b}_1(t) + c_1 \bar{d}_1(t), \ldots, \bar{b}_\rho(t) + c_\rho \bar{d}_\rho(t), \bar{b}_{\rho+1}(t), \ldots, \bar{b}_m(t), \eta); \\ &\eta(0) = \eta_0. \end{aligned} \tag{17}$$

Let us prove that the parameter $c_* \in R^\rho$ exists, and the solution $\eta(t, c)$ of Cauchy problem (17) satisfies the condition $\eta(t_*, c_*) = \eta_*$. If $b_i(t)$, $d_i(t) \in C^{r_i}([0, t_*])$, $i = \overline{1, m}$, and $q(z^1, \ldots, z^m, \eta) \in C^\infty(R^n)$, then the vector-function $\eta(t, c)$ is differentiable using the parameter c, while the matrix function $\nu = \partial\eta/\partial c$ satisfies the system of equations (13):

$$\dot{\nu} = K\nu + \sum_{i=1}^{r} \left(A_i + \frac{\partial p}{\partial z_i} \right) d^{(i-1)}(t); \qquad \nu(0) = 0, \tag{18}$$

which is obtained as the result of system (17) differentiation with the parameter c.

Now we introduce the mapping $\Psi : R^\rho \to R^\rho$, which assigns to each parameter $c \in R^\rho$, the value $\eta(t_*, c) \in R^\rho$ of the $\eta(t, c)$ solution of Cauchy problem (17) at a moment of time t_*. Let us show that while satisfying the theorem's conditions, there exists the parameter c_*, for which the equality $\Psi(c_*) = \eta_*$ is fulfilled. For this, we will introduce the mapping $v : R^\rho \to R^\rho$, functioning according to the rule

$$v(c) = c - M^{-1}(\Psi(c) - \eta_*).$$

The equality $\Psi(c_*) = \eta_*$ is equivalent to the fact that the parameter c_* is a fixed point of the mapping v. To prove the existence of the fixed point in the mapping v, we prove that the mapping v is compressing. The Jacobi matrix of the mapping v has a form of $v'(c) = E - M^{-1}\Psi'(c)$, where E is a unity $\rho \times \rho$-matrix; $\Psi'(c)$ is the Jacobi matrix of the mapping Ψ. According to the definition of the mapping Ψ, $\Psi'(c) = \nu(t_*)$, then

$$v'(c) = E - M^{-1}\nu(t_*).$$

Let us denote $D(t) = \int\limits_0^t d(\tau)d\tau$. The choice of the functions $d(t)$ in formula (9) ensures that with $t \in [0, t_*]$ both the inequality $0 \leqslant D(t) \leqslant 1$ and the equality $D(t_*) = 1$ are fulfilled. Let us consider the matrix function

$$W(t) = D(t)E + \sum_{i=1}^{r-1} d^{(i-1)}(t) N_i - M^{-1}\nu(t), \tag{19}$$

here N_i is $\rho \times \rho$-matrices, which will be chosen later. From the equations $D(0) = 0$, $d(0) = 0, \ldots, d^{(r-2)}(0) = 0$, $\nu(0) = 0$, it follows that $W(0) = 0$, and from the equations $D(t_*) = 1$, $d(t_*) = 0, \ldots, d^{(r-2)}(t_*) = 0$ – $W(t_*) = E - M^{-1}\nu(t_*)$. Having shown that $\|W(t_*)\| \leqslant \gamma < 1$, thereby we can write the inequality $\|\upsilon'(c)\|\leqslant\gamma < 1$. Therefore, we can prove that the mapping υ is compressing. We calculate $\dot{W}$ with the help of (18):

$$\dot{W} = d(t)E + \sum_{i=2}^{r} d^{(i-1)}(t)N_{i-1} - M^{-1}\dot{\nu} =$$
$$= d(t)E + \sum_{i=2}^{r} d^{(i-1)}(t)N_{i-1} - M^{-1}\left[K\nu + \sum_{i=1}^{r}\left(A_i + \frac{\partial p}{\partial z_i}\right)d^{(i-1)}(t)\right]. \quad (20)$$

Taking (19) into consideration, we express $\nu(t)$ in terms of $W(t)$:

$$\nu(t) = M\left(D(t)E + \sum_{i=1}^{r-1} d^{(i-1)}(t)N_i - W(t)\right),$$

substitute the received relation in (20). As a result, we have the equality

$$\begin{aligned}\dot{W} &= PW + [E - M^{-1}A_1 - PN_1]d(t) + \\ &+ \sum_{i=2}^{r-1}[N_{i-1} - M^{-1}A_i - PN_i]d^{(i-1)}(t) + \\ &+ [N_{r-1} - M^{-1}A_r]d^{(r-1)}(t) - M^{-1}\sum_{i=1}^{r}\frac{\partial p}{\partial z_i}d^{(i-1)}(t) - PD(t).\end{aligned} \quad (21)$$

Now we choose matrices $N_1, \ldots, N_{r-1}$ from the condition

$$\begin{aligned}&E - M^{-1}A_1 - PN_1 = 0;\\ &N_{i-1} - M^{-1}A_i - PN_i = 0, \quad i = \overline{2, r-1};\\ &N_{r-1} - M^{-1}A_r = 0.\end{aligned} \quad (22)$$

With a direct substitution, we can show that the solution to system (22) of the matrix equations is the matrices

$$\begin{aligned}&N_1 = M^{-1}(A_2 + KA_3 + \ldots + K^{r-2}A_r);\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &N_{r-2} = M^{-1}(A_{r-1} + KA_r);\\ &N_{r-1} = M^{-1}A_r.\end{aligned} \quad (23)$$

Let us assume that the matrices N_i are specified by formulae (23). Then equality (21) will take the form of

$$\dot{W} = PW + R(t),$$

where

$$R(t) = -M^{-1}\sum_{i=1}^{r}\frac{\partial p}{\partial z_i}d^{(i-1)}(t) - PD(t).$$

In accordance with condition 3 of the theorem, for all $y \in R^\rho$ the inequality $(Py, y) \leqslant \lambda \|y\|^2$ is fulfilled, therefore, according to Lemma 1, the solution $W(t)$ of the Cauchy problem

$$\dot{W} = PW + R(t), \quad W(0) = 0$$

satisfies the inequality

$$\|W(t_*)\| \leqslant e^{\lambda t_*} \int\limits_0^{t_*} \|R(t)\| e^{-\lambda t} dt. \tag{24}$$

Using the triangle inequality, condition 3 of the theorem and the inequality $0 \leqslant D(t) \leqslant 1$, fulfilled with all$t \in [0, t_*]$, we obtain

$$\|R(t)\| = \left\| M^{-1} \sum_{i=1}^{r} \frac{\partial p}{\partial z_i} d^{(i-1)}(t) + PD(t) \right\| \leqslant$$
$$\leqslant \|M^{-1}\| \sum_{i=1}^{r} \left\| \frac{\partial p}{\partial z_i} \right\| |d^{(i-1)}(t)| + \|P\| D(t) \leqslant$$
$$\leqslant \|M^{-1}\| \sum_{i=1}^{r} \varepsilon |d^{(i-1)}(t)| + \|P\| \leqslant \|M^{-1}\| \varepsilon L + \|P\|.$$

With regard to this estimation and notation (16), inequality (24) is taking the form of

$$\|W(t_*)\| \leqslant e^{\lambda t_*} \int\limits_0^{t_*} (\|M^{-1}\| \varepsilon L + \|P\|) e^{-\lambda t} dt = \gamma.$$

If $\gamma < 1$, then $\|v'(c)\| = \|W(t_*)\| \leqslant \gamma < 1$ and, consequently, the mapping v is compressing. Thus, if the theorem conditions are satisfied, the mapping v is compressing and has a fixed point c_*. With $c_1 = c_{1*}, \ldots, c_\rho = c_{\rho *}$, $c_{\rho+1} = 0, \ldots, c_m = 0$, the solution $\eta(t)$ of Cauchy problem (5) satisfies the condition $\eta(t_*) = \eta_*$. The functions

$$B_1(t) = b_1(t) + c_{1*} d_1(t), \ldots, B_\rho(t) = b_\rho(t) + c_{\rho *} d_\rho(t),$$

$$B_{\rho+1} = b_{\rho+1}(t), \ldots, B_m(t) = b_m(t)$$

satisfy all the conditions of theorem 2, hence terminal problem (3), (4) for system (2) has got a solution.

Numerical procedure. The method of construction of terminal problem (3) solution, (4) for system (2) results from theorem 3 proving. Let us take a random number $c^{(0)} \in R^\rho$ and build a sequence of approximations $\{c^{(j)}\}$ according to the rule

$$c^{(j+1)} = c^{(j)} - M^{-1}(\Psi(c^{(j)}) - \eta_*), \quad j = 0, 1, \ldots \tag{25}$$

In order to determine the value $\Psi(c^{(j)})$, it is necessary to find the solution $\eta(t, c^{(j)})$ to the Cauchy problem

$$\dot{\eta} = q(\overline{b}_1(t) + c_1^{(j)}\overline{d}_1(t), \ldots, \overline{b}_\rho(t) + c_\rho^{(j)}\overline{d}_\rho(t), \overline{b}_{\rho+1}(t), \ldots, \overline{b}_m(t), \eta);$$
$$\eta(0) = \eta_0.$$

Then $\Psi(c^{(j)}) = \eta(t_*, c^{(j)})$.

As the mapping v is compressing, the sequence $\{c^{(j)}\}$ converges to the fixed point c_* of the mapping v. Thereby, the estimation is true

$$\|c^{(j)} - c_*\| \leqslant \frac{\gamma^j}{1-\gamma}\|c^{(1)} - c^{(0)}\|. \tag{26}$$

It follows from (25) that

$$\Psi(c^{(j)}) - \eta_* = M(c^{(j+1)} - c^{(j)}),$$

hence, using the triangle inequality and estimation (26), we obtain

$$\|\Psi(c^{(j)}) - \eta_*\| \leqslant \|M\|\|c^{(j+1)} - c^{(j)}\| = \|M\|\|c^{(j+1)} - c_* + c_* - c^{(j)}\| \leqslant$$
$$\leqslant \|M\|\|c^{(j+1)} - c_*\| + \|M\|\|c_* - c^{(j)}\| \leqslant \frac{\|M\|}{1-\gamma}\left(\gamma^{j+1} + \gamma^j\right)\|c^{(1)} - c^{(0)}\| =$$
$$= \frac{(1+\gamma)\gamma^j}{1-\gamma}\|M\|\|c^{(1)} - c^{(0)}\|.$$

Having chosen the number J from the condition

$$\frac{(1+\gamma)\gamma^J}{1-\gamma}\|M\|\|c^{(1)} - c^{(0)}\| \leqslant \sigma,$$

where $\sigma > 0$ is a given accuracy, we try to obtain the inequality fulfilment

$$\|\Psi(c^{(J)}) - \eta_*\| \leqslant \sigma. \tag{27}$$

The vector-functions

$$z^1 = \overline{b}_1(t) + c_1^{(J)}\overline{d}_1(t), \ldots, z^\rho = \overline{b}_\rho(t) + c_\rho^{(J)}\overline{d}_\rho(t),$$

$$z^{\rho+1} = \overline{b}_{\rho+1}(t), \ldots, z^m = \overline{b}_m(t); \quad \eta = \eta(t, c^{(J)}), \quad t \in [0, t_*],$$

specify the t-parameter curve in the range of system (2) conditions, connecting states (3) and (4). The control implementing this trajectory as system (2) trajectory, can be found using formula (7), if we assume that

$$\overline{B}_1(t) = \bar{b}_1(t) + c_1^{(J)}\overline{d}_1(t), \ldots, \overline{B}_\rho(t) = \bar{b}_\rho(t) + c_\rho^{(J)}\overline{d}_\rho(t),$$

$$\overline{B}_{\rho+1} = \bar{b}_{\rho+1}(t), \ldots, \overline{B}_m(t) = \bar{b}_m(t); \quad \eta(t) = \eta(t, c^{(J)}).$$

Example. Let us consider the system

$$\begin{aligned} &\dot{z}_1^i = z_2^i; \\ &\dot{z}_2^i = u_i, \qquad i = 1, 2; \\ &\dot{\eta}_1 = -0.1\eta_2 + z_1^1 + z_2^2 + 0.08\cos z_2^1; \\ &\dot{\eta}_2 = 0.1\eta_1 + z_1^2 + z_2^1 - 0.08\sin z_2^2 \end{aligned} \tag{28}$$

with the following boundary conditions:

$$\begin{aligned} &z_1^1(0) = 0, \quad z_2^1(0) = 0, \quad z_1^2(0) = 0, \; z_2^2(0) = 0, \;\; \eta_1(0) = 0, \;\; \eta_2(0) = 0, \\ &z_1^1(2) = -4, \; z_2^1(2) = -8, \; z_1^2(2) = 0, \; z_2^2(2) = 4, \; \eta_1(2) = -5, \; \eta_2(2) = 4. \end{aligned}$$

For this task $t_* = 2$, $m = 2$, $\rho = 2$, $r_1 = r_2 = 2$, $z_1 = (z_1^1, z_1^2)^{\mathrm{T}}$, $z_2 = (z_2^1, z_2^2)^{\mathrm{T}}$,

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & -0,1 \\ 0,1 & 0 \end{pmatrix},$$

$$p(z^1, z^2) = \begin{pmatrix} 0.08\cos z_2^1 \\ -0.08\sin z_2^2 \end{pmatrix}, \quad M = A_1 + KA_2 = \begin{pmatrix} 0.9 & 0 \\ 0 & 1.1 \end{pmatrix},$$

$$M^{-1} = \begin{pmatrix} 10/9 & 0 \\ 0 & 10/11 \end{pmatrix}, \quad P = M^{-1}KM = \begin{pmatrix} 0 & -11/90 \\ 9/110 & 0 \end{pmatrix},$$

$$\frac{\partial p}{\partial z_1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \frac{\partial p}{\partial z_2} = \begin{pmatrix} -0.08\sin z_2^1 & 0 \\ 0 & -0.08\cos z_2^2 \end{pmatrix}.$$

Since $\|\partial p/\partial z_1\| = 0$, a $\|\partial p/\partial z_2\| \leqslant 0.08\sqrt{2}$, we can assume $\varepsilon = 0.08\sqrt{2}$ as the number ε from condition 3 of theorem 3. Matrix $(P + P^{\mathrm{T}})/2$ has the form of

$$\frac{1}{2}(P + P^{\mathrm{T}}) = \begin{pmatrix} 0 & -2/99 \\ -2/99 & 0 \end{pmatrix},$$

Its largest proper value $\lambda = 2/99$.

Let us check the fulfilment of theorem 3 conditions. The function $d(t)$, built with formula (9), has the form of $d(t) = \dfrac{15}{16}t^2(2-t)^2$, therefore,

$$d'(t) = \frac{15}{4}t(t-1)(t-2); \quad L = \max_{[0,2]}\{d(t) + |d'(t)|\} \leqslant 1.95.$$

In connection with the fact that $\gamma = (\|M^{-1}\|\varepsilon L + \|P\|)\dfrac{e^{\lambda t_*} - 1}{\lambda} \approx$ $\approx 0.947 < 1$, the condition of theorem 3 has been satisfied and the terminal problem under consideration has got a solution.

Now we select the function $b_1(t) = -t^3 + t^2$ as the function $b_1(t)$, which satisfies the conditions

$$b_1(0) = 0,\ b_1'(0) = 0,\ b_1(2) = -4,\ b_1'(2) = -8,$$

and the function $b_2(t) = t^3 - 2t^2$ as the function $b_2(t)$, which satisfies the conditions

$$b_2(0) = 0,\ b_2'(0) = 0,\ b_2(2) = 0,\ b_2'(2) = 4.$$

Let us specify the initial approximation for the vector of parameters $c^{(0)} = (0; 0)^{\mathrm{T}}$ and the accuracy $\sigma = 0.001$. Let us build the sequence of approximations $\{c^{(j)}\}$ using formula (25), assuming that $\eta_* = (-5; 4)^{\mathrm{T}}$, $\Psi(c^{(j)}) = \eta(t_*, c^{(j)})$, where $\eta(t, c^{(j)}) = (\eta_1(t, c^{(j)}), \eta_2(t, c^{(j)}))^{\mathrm{T}}$, which is the solution to the Cauchy problem:

$$\begin{aligned}
&\dot\eta_1 = -0.1\eta_2 + b_1(t) + c_1^{(j)} d(t) + b_2'(t) + \\
&\qquad\qquad + c_2^{(j)} d'(t) + 0.08\cos(b_1'(t) + c_1^{(j)} d'(t);\\
&\dot\eta_2 = 0.1\eta_1 + b_2(t) + c_2^{(j)} d(t) + b_{1'}(t) + \\
&\qquad\qquad + c_1^{(j)} d'(t) - 0.08\sin(b_2'(t) + c_2^{(j)} d'(t);\\
&\eta_1(0) = 0,\ \eta_2(0) = 0,
\end{aligned}$$

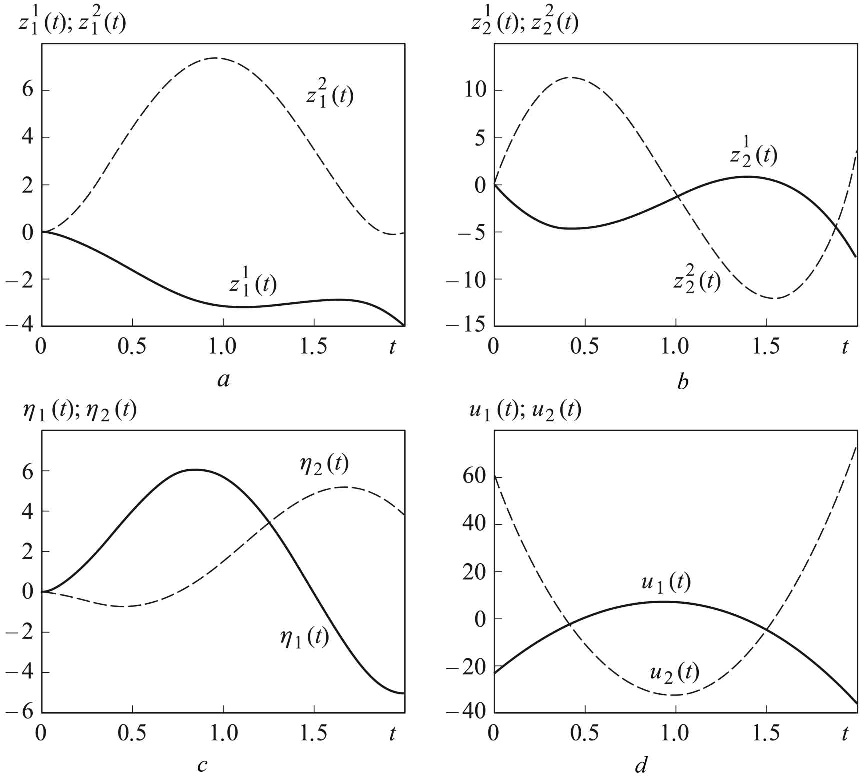
being determined on each iteration using the Runge – Kutta method of the fourth order. The calculations showed that inequality (27) can be fulfilled with $J = 6$, hence the point $c^{(6)} = (-3.287; 8.933)^{\mathrm{T}}$ is the fixed point of the mapping v with the accuracy σ. The functions

$$z_1^1 = b_1(t) + c_1^{(6)} d(t),\ z_2^1 = b_1'(t) + c_1^{(6)} d'(t),\ z_1^2 = b_2(t) + c_2^{(6)} d(t),$$

$$z_2^2 = b_2'(t) + c_2^{(6)} d'(t);\ \eta_1 = \eta_1(t, c^{(6)}),\ \eta_2 = \eta_2(t, c^{(6)})$$

specify the t-parameter curve in the range of system (28) conditions, connecting the initial and final system statuses. The controls $u_1 = b_1''(t) +$ $+ c_1^{(6)} d''(t)$, $u_2 = b_2''(t) + c_2^{(6)} d''(t)$ realize this curve as the trajectory of system (28) and are a solution of the terminal problem under consideration. Functional relations $z_1^1(t)$, $z_1^2(t)$, $z_2^1(t)$, $z_2^2(t)$, $\eta_1(t)$, $\eta_2(t)$, $u_1(t)$, $u_2(t)$ are shown in the picture.

Conclusion. The terminal problem for the affine systems, which are not linearizable by a feedback, is considered. It is supposed that using a smooth nondegenerate change of variables within the range of states,



Functions $z_1^1(t)$, $z_1^2(t)$ (*a*), $z_2^1(t)$, $z_2^2(t)$ (*b*), $\eta_1(t)$, $\eta_2(t)$ (*c*), and $u_1(t)$, $u_2(t)$ (*d*)

the system can be transformed to a regular quasicanonical form. Along with this, the terminal problem for the initial system is transformed to the equivalent terminal problem for the system of a quasicanonical form. For a quasicanonical system, the sufficient condition of the existence of the terminal problems solution is proved. A method for solving the terminal tasks is proposed on the basis of this condition. An example is given of the terminal task solution development using the method proposed for the sixth-order system.

The work was supported by the Russian Foundation for Basic Research (grants 14-07-00813, 13-07-00736).

REFERENCES

[1] Krasnoshchechenko V.I., Krishchenko A.P. Nelineynye sistemy: geometricheskie metody analiza i sinteza [Nonlinear systems: geometric methods for analysis and synthesis]. Moscow, MGTU im. N.E. Baumana Publ., 2005. 520 p.

[2] Elkin V.I. Reduktsiya nelineynykh upravlyaemykh sistem: differentsial'no geometricheskiy podkhod [Reduction of nonlinear controlled systems: a differential geometric approach]. Moscow, Nauka Publ., 1997. 320 p.

[3] Agrachev A.A., Sachkov Yu.L. Geometricheskaya teoriya upravleniya [Geometric control theory]. Moscow, Fizmatlit Publ., 2005. 392 p.

[4] Zhevnin A.A., Krishchenko A.P. Upravlyaemost' nelineinykh system i sintez algoritmov upravlenia [Controllability of nonlinear systems and synthesis of control algorithms]. *Dokl. Akad. Nauk SSSR* [Proc. Acad. Sci. USSR], 1981, vol. 258, no. 3, pp. 805–809 (in Russ.).
[5] Sachkova E.F. Priblizhennoye reshenie dvukhtochechnykh granichnykh zadach dlya system s lineinymi upravleniami [Approximative solution of two-point boundary value problems for systems with linear control]. *Avtom. Telemekh.* [Automation and Remote Control], 2009, no. 4, pp. 179–189 (in Russ.).
[6] Emel'yanov S.V., Krishchenko A.P., Fetisov D.A. Issledovanie upravliaemosti affinnykh sistem [The study of controllability of affine systems]. *Dokl. RAN* [Proc. Russ. Acad. Sci.], 2013, vol. 449, no. 1, pp. 15–18 (in Russ.).
[7] Krishchenko A.P., Fetisov D.A. Preobrazovanie affinnykh system i reshenie zadach terminal'nogo upravlenia [Transformation of affine systems and solving problems of termination control]. *Vestn. Mosk. Gos. Tekh. Univ. im. N.E. Baumana, Estestv. Nauki* [Herald of Bauman Moscow State Tech. Univ., Nat. Sci.], 2013, no. 2, pp. 3–16 (in Russ.).
[8] Krishchenko A.P., Fetisov D.A. Terminal'naya zadacha dlya nogomernykh affinnykh system [Terminal problem for multidimensional affine systems]. *Dokl. RAN* [Proc. Russ. Acad. Sci.], 2013, vol. 452, no. 2, pp. 144–149 (in Russ.).
[9] Fetisov D.A. Ob odnom metode reshenia terminal'nykh zadach dlya affinnykh system [On a method for solving terminal control problems for affine systems]. E*lektr. Nauchno-Tehn. Izd. "Nauka i obrazovanie", MGTU im. N.E. Baumana* [El. Sc.-Tech. Publ. "Science and Education" of BMSTU], 2013, no. 11. Available at: http:// technomag.bmstu.ru/doc/622543.html (accessed 20.11.2013) (in Russ.). DOI: 10.7463/1113.0622543
[10] Krut'ko P.D. Obratnye zadachi dinamiki upravlyaemykh sistem. Nelineynye modeli [Inverse problems for dynamics controlled systems. Nonlinear models]. Moscow, Nauka Publ., 1988. 328 p.
[11] Krishchenko A.P., Klinkovskiy M.G. Preobrazovanie affinnykh system s upravleniem i zadacha stabilizatsii [Transformation of affine systems with control and the problem of stabilization]. *Differ. Uravn.* [Differ. Equations], 1992, vol. 28, no. 11, pp. 1945–1952 (in Russ.).
[12] Hartman Ph. Ordinary Differential Equations, 1st ed. N.Y., J. Wiley, 1964. (Russ. Ed.: Khartman F. Obyknovennye differentsial'nye uravneniya. Moscow, Mir Publ., 1970. 720 p.).
[13] Filippov A.F. Vvedenie v teoriyu differentsial'nykh uravneniy [Introduction to the differential equations theory]. Moscow, Editorial URSS Publ., 2004. 240 p.

The original manuscript was received by the editors on 18.04.2014

Contributor

D.A. Fetisov — Ph.D. (Phys.-Math.), Associate Professor, Department of Mathematical Simulation, Bauman Moscow State Technical University, author of 15 research publications in the field of mathematical control theory.
Bauman Moscow State Technical University, 2-ya Baumanskaya ul. 5, Moscow, 105005 Russian Federation.

The translation of this article from Russian into English is done by O.G. Rumyantseva, Senior Lecturer, Linguistics Department, Bauman Moscow State Technical University under the general editorship of N.N. Nikolaeva, Ph.D. (Philol.), Associate Professor, Linguistics Department, Bauman Moscow State Technical University.