MATHEMATICS

SOLUTION TO THE MIXED BOUNDARY-VALUE PROBLEM FOR LAPLACE EQUATION IN MULTIDIMENTIONAL INFINITE LAYER

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The paper considers the solution to the mixed boundary-value problem of finding a harmonic function of n variables for the domain confined by two parallel hyper-planes. This function was determined by its values on a hyper-plane and its normal derivative on another hyper-plane. The obtained solution is presented as a sum of two integrals which kernels are expressed only in terms of elementary functions in the case of the even-dimension space. In contrast to the odd-dimension space they are also expressed through the Bessel functions. If the given boundary values are tempered distributions, then the solution is written as a convolution of the kernels with these functions. The opportunity of practical application of the obtained formulas is illustrated by the example of forming up a filtration flow under the spot dam with a aquiclude.

***Keywords*:** harmonic functions, Fourier transform, tempered distributions, filtration theory.

Harmonic functions of two and three variables describe many stationary processes of underground hydrodynamics, thermal conductivity, etc. Therefore, the search for solutions to various boundary value problems for the Laplace equation (and new simpler forms of solutions) is highly relevant. In case of simply connected planar domains (e.g., bandwidth) solving these problems by means of conformal transformations is reduced to solving them for canonical domains — a circle and a half-plane. If there are more than two variables, it is not possible. A unique solution has to be seached for each domain. In case of a half-space the primary method of solving the boundary value problems for linear partial differential equations with constant coefficients is Fourier transformation for the variables in the boundary hyperplane [1]. Poisson and Neumann kernels of integrals which solve both the first (Dirichlet problem) and second (Neumann problem) boundary value problems for the Laplace equation in a half-space are well known [2, 3]. In case of the bandwidth and infinite layer in three-dimensional space the kernels of the integral representing various solutions of boundary value problems for the Laplace equation can be obtained by the method of reflections in the form of sums of infinite series [4, 5]. For an infinite layer in n-dimensional space the first boundary value problem was solved by one of the authors of [6]. Wherein it was managed to sum up the above mentioned series, expressing their sum in terms of elementary functions. In this paper the mixed boundary value Dirichlet — Neumann

problem for the Laplace equation was solved for an infinite layer in n-dimensional space. The solutions of this problem for the bandwidth and the infinite layer in three-dimensional space obtained by the method of reflection are known [5]. In this work, the problem has been solved by the Fourier transform method. The recurrence relation between the integral kernels for n- dimensional and $(n + 2)$-dimensional layers was obtained and for $n = 2, 3, 4$ these kernels are expressed in terms of elementary functions (for $n = 3$ Bessel functions are also used).

Notations. Statement of the problem. Let us introduce the following notation

$$x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \quad (x, y) = (x_1, \ldots, x_n, y) \in \mathbb{R}^{n+1}, \quad y \in \mathbb{R};$$

$$|x| = \sqrt{x_1^2 + \ldots + x_n^2}, \quad \langle x, t \rangle = x_1 t_1 + \cdots + x_n t_n, \quad dx = dx_1 \cdots dx_n;$$

$$\Delta u (x, y) = \Delta u = u_{x_1 x_1} + \ldots + u_{x_n x_n} + u_{yy}$$

— Laplacian;

$$F(t) = \mathcal{F}[f](t) = \int\limits_{\mathbb{R}^n} f(x) e^{i\langle x, t \rangle} dx$$

— Fourier transform method of a summable function $f(x)$. If the function $f(x, y)$ summable on x depends on the variables x and y, then its Fourier transform on x will be denoted as

$$\mathcal{F}_x [f](t, y) = \int\limits_{\mathbb{R}^n} f(x, y) e^{i\langle x, t \rangle} dx.$$

The inverse Fourier transform of a summable function $F(t)$ is defined similarly

$$f(x) = \mathcal{F}^{-1}[F](x) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} F(t) e^{-i\langle x, t \rangle} dt,$$

as well as the Fourier transform of the function $F(t, y)$ summable on t

$$\mathcal{F}_t^{-1}[F](x, y) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} F(t, y) e^{-i\langle x, t \rangle} dt.$$

The definition of the Fourier transform of tempered distribution is given in [7].

Let us consider the mixed boundary value problem: to find a function $u(x, y)$ of $n + 1$ variable harmonic in

$$D = \{(x, y) : 0 < y < a\} \subset \mathbb{R}^{n+1};$$

$$\Delta u\,(x,y) = 0, \quad x \in \mathbb{R}^n, \quad 0 < y < a \,,$$

in order to satisfy the boundary conditions

$$u\,(x,0) = \varphi\,(x)\,, \quad x \in \mathbb{R}^n;$$

$$u_y\,(x,a) = \psi\,(x)\,, \quad x \in \mathbb{R}^n.$$

Solution of the problem for the general case. Derivation of the recurrence relation. Let us apply the Fourier transform on x to the Laplace equation, denoting

$$U\,(t,y) = \mathcal{F}_x\,[u]\,(t,y)\,; \quad \Phi\,(t) = \mathcal{F}\,[\varphi]\,(t)\,; \quad \Psi\,(t) = \mathcal{F}[\psi](t).$$

We will obtain a boundary value problem for an ordinary differential equation with a parameter $t \in \mathbb{R}^n$:

$$- |t|^2\,U\,(t,y) + U_{yy}\,(t,y) = 0;$$

$$U\,(t,0) = \Phi\,(t)\,, \quad U_y\,(t,a) = \Psi\,(t)\,.$$

The solution to this boundary value problem is a function

$$U\,(t,y) = \Phi\,(t)\,\frac{\mathrm{ch}(|t|\,(a-y))}{\mathrm{ch}(a|t|)} + \Psi\,(t)\,\frac{\mathrm{sh}\,(|t|\,y)}{|t|\,\mathrm{ch}\,(a\,|t|)}\,\,.$$

Applying the inverse Fourier transform, we are finding the solution to the original mixed boundary value problem as a convolution

$$u\,(x,y) = \varphi\,(x) * K_n\,(x,y) + \psi\,(x) * L_n\,(x,y)\,, \tag{1}$$

where the kernels are designated as

$$K_n\,(x,y) = \mathcal{F}_t^{-1}\,[k_n]\,(x,y)\,, \quad k_n\,(|t|\,,y) = \frac{\mathrm{ch}\,(|t|\,(a-y))}{\mathrm{ch}(a|t|)}, \quad t \in \mathbb{R}^n;$$

$$L_n\,(x,y) = \mathcal{F}_t^{-1}\,[l_n]\,(x,y)\,, \quad l_n(|t|\,,y) = \frac{\mathrm{sh}\,(|t|\,y)}{|t|\,\mathrm{ch}\,(a\,|t|)}, \quad t \in \mathbb{R}^n.$$

Since

$$\frac{\partial}{\partial y} l_n\,(|t|\,,y) = k_n\,(|t|\,,a-y)\,,$$

then

$$\frac{\partial}{\partial y}\,L_n\,(x,y) = K_n\,(x,a-y)$$

and $\dfrac{\partial}{\partial y}\,L_n\,(x,y)$ as $y \to a - 0$, behaves in the same way as $K_n\,(x,y)$ if $y \to +0$.

It is easy to notice that $k_n(|t|, y)$ and $l_n(|t|, y)$ are infinitely differentiable and rapidly decreasing functions $t \in \mathbb{R}^n$, i.e. they belong to the space $\mathcal{S}(\mathbb{R}^n)$ [7]. These functions are also spherically symmetric, so while calculating the inverse Fourier transform it is possible to pass to the spherical coordinates in the space $\mathbb{R}^n$. Let us denote $|x| = r$, $|t| = \rho$, σ_{n-1} as an area of the unit sphere in the space $\mathbb{R}^n$. For any spherically symmetric function $h_n(|t|) = h_n(\rho) \in \mathcal{S}(\mathbb{R}^n)$ we have

$$H_n(|x|) = H_n(r) = \mathcal{F}_t^{-1}[h_n](x) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} h_n(|t|) e^{-i\langle x, t\rangle} dt =$$

$$= \frac{\sigma_{n-1}}{(2\pi)^n} \int\limits_0^\infty h_n(\rho) \rho^{\frac{n}{2}} d\rho \int\limits_0^\pi e^{-ir\rho\cos\theta} \sin^{n-2}\theta d\theta =$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}} r^{\frac{n}{2}-1}} \int\limits_0^\infty h_n(\rho) \rho^{\frac{n}{2}} J_{\frac{n}{2}-1}(r\rho) d\rho,$$

where $J_{\frac{n}{2}-1}(r\rho)$ is the Bessel function of the first kind of order $\nu = n/2 - 1$ [7, 8]. The above formula is valid and it is easy to be checked.

By differentiating the above equation and by considering the formula of the theory of Bessel functions [4]

$$\frac{\nu J_\nu(r\rho)}{r\rho} - J'_\nu(r\rho) = J_{\nu+1}(r\rho),$$

we obtain the recurrence formula

$$H_{n+2}(r) = -\frac{1}{2\pi r} \frac{\partial}{\partial r} H_n(r).$$

For this formula it is necessary to know the kernels $K_n(x, y)$ and $L_n(x, y)$ only for $n = 1$ and $n = 2$.

Since the Fourier transform moves the space $\mathcal{S}(\mathbb{R}^n)$ into itself, the kernels $K_n(x, y) = K_n^*(|x|, y)$ and $L_n(x, y) = L_n^*(|x|, y)$ when $\forall y \in (0, a)$, are spherically symmetric functions of x in the space $\mathcal{S}(\mathbb{R}^n)$. Therefore, the convolution (1) exists for any tempered distribution $\varphi(x) \in \mathcal{S}'(\mathbb{R}^n)$, $\psi(x) \in \mathcal{S}'(\mathbb{R}^n)$, and can be written as

$$\varphi(x) * K_n(x, y) + \psi(x) * L_n(x, y) =$$
$$= (\varphi(t), K_n(x-t, y)) + (\psi(t), L_n(x-t, y)).$$

When $\varphi(x)$ and $\psi(x)$ are normal functions of polynomial growth, convolution (1) can be written as a sum of integrals

$$\int\limits_{\mathbb{R}^n} \varphi(t) K_n(x-t, y) dt + \int\limits_{\mathbb{R}^n} \psi(t) L_n(x-t, y) dt.$$

It is easy to test the feasibility of the following properties in the area D:

1) $K_n(x, y) > 0$;

2) $\displaystyle\int\limits_{\mathbb{R}^n} K_n(x, y)\; dx = 1$;

3) when $\forall \delta > 0$, $\lim\limits_{y \to +0} \sup\limits_{|x| \geq \delta} K_n(x, y) = 0$.

These properties mean that $K_n(x, y)$ is an *approximate identity*, or δ-shaped system of functions of x (with parameter y). When $y \to +0$, $K_n(x, y)$ converges weakly to a δ-function $\delta(x)$. Since

$$\frac{\partial}{\partial y} L_n(x, y) = K_n(x, a - y),$$

$\dfrac{\partial}{\partial y} L_n(x, y)$ is an approximate identity as $y \to a - 0$.

Therefore, if $\varphi(x) \in \mathcal{S}'(\mathbb{R}^n)$ and $\psi(x) \in \mathcal{S}'(\mathbb{R}^n)$, then the formula (1) gives a generalized solution to the problem: $\lim\limits_{y \to +0} u(x, y) = \varphi(x)$ in $\mathcal{S}'$; $\lim\limits_{y \to a - 0} u_y(x, y) = \psi(x)$ in $\mathcal{S}'$.

If the functions $\varphi(x)$ and $\psi(x)$ are normal functions of polynomial growth, than at each point of continuity

$$\lim\limits_{y \to +0} u(x, y) = \varphi(x); \quad \lim\limits_{y \to a - 0} u_y(x, y) = \psi(x).$$

Solution to the mixed boundary value problem for the bandwidth on the plane. In the case of two variables in view of parity functions in t $(k_1(|t|, y) = k_1(t, y), l_1(|t|, y) = l_1(t, y))$, the inverse Fourier transform can be found from the tables [9]:

$$\mathcal{F}_t^{-1}[k_1](x, y) = \frac{1}{a} \frac{\sin\left(\dfrac{\pi y}{2a}\right) \operatorname{ch}\left(\dfrac{\pi x}{2a}\right)}{\operatorname{ch}\left(\dfrac{\pi x}{a}\right) - \cos\left(\dfrac{\pi y}{a}\right)} = K_1(x, y);$$

$$\mathcal{F}_t^{-1}[l_1](x, y) = \frac{1}{2\pi} \ln\left(\frac{\operatorname{ch}\left(\dfrac{\pi x}{2a}\right) + \sin\left(\dfrac{\pi y}{2a}\right)}{\operatorname{ch}\left(\dfrac{\pi x}{2a}\right) - \sin\left(\dfrac{\pi y}{2a}\right)}\right) = L_1(x, y).$$

If the functions $\varphi(x)$ and $\psi(x)$ are the functions of polynomial growth, then the solution to the mixed problem is written as the integral formula

$$u(x, y) = \frac{1}{a} \sin\left(\frac{\pi y}{2a}\right) \int\limits_{-\infty}^{\infty} \varphi(t) \frac{\operatorname{ch}\left(\dfrac{\pi(x - t)}{2a}\right)}{\operatorname{ch}\left(\dfrac{\pi(x - t)}{a}\right) - \cos\left(\dfrac{\pi y}{a}\right)} dt +$$

$$+ \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \psi(t) \ln \left(\frac{\operatorname{ch}\left(\frac{\pi(x-t)}{2a} \right) + \sin\left(\frac{\pi y}{2a} \right)}{\operatorname{ch}\left(\frac{\pi(x-t)}{2a} \right) - \sin\left(\frac{\pi y}{2a} \right)} \right) dt \ .$$

The solution to this problem is known, but the one with the kernel in the form of an infinite series [5]:

$$u(x, y) = \int\limits_{-\infty}^{\infty} \varphi(t) \left[\frac{\partial}{\partial \tau} G(x, y, t, \tau) \right]_{\tau=0} dt + \int\limits_{-\infty}^{\infty} \psi(t) G(x, y, t, a) \, dt,$$

where

$$G(x, y, t, \pi) =$$
$$= \frac{1}{a} \sum\limits_{0}^{\infty} \frac{1}{\mu_n} \exp\left(-\mu_n |x - t| \right) \sin(\mu_n y) \ \sin(\mu_n \tau), \quad \mu_n = \frac{\pi(2n+1)}{2a}.$$

Solution to the mixed boundary value problem for an infinite layer in three-dimensional space. For three variables kernels can not be expressed in terms of elementary functions:

$$K_2(x, y) = \frac{1}{2\pi} \int\limits_{0}^{\infty} \frac{\operatorname{ch}(\rho(a-y))}{\operatorname{ch}(a\rho)} \rho J_0\left(\rho |x| \right) d\rho;$$

$$L_2(x, y) = \frac{1}{2\pi} \int\limits_{0}^{\infty} \frac{\operatorname{sh}(\rho y)}{\operatorname{ch}(a\rho)} J_0\left(\rho |x| \right) d\rho.$$

If the functions $\varphi(x)$ and $\psi(x)$ are the functions of polynomial growth, then the solution of the mixed problem is written as the integral formula

$$u(x, y) = \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \varphi(t) \, dt \int\limits_{0}^{\infty} \frac{\operatorname{ch}(\rho(a-y))}{\operatorname{ch}(a\rho)} \rho J_0\left(\rho |x-t| \right) d\rho +$$
$$+ \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \psi(t) \, dt \int\limits_{0}^{\infty} \frac{\operatorname{sh}(\rho y)}{\operatorname{ch}(a\rho)} J_0\left(\rho |x-t| \right) d\rho.$$

In this case the solution to the problem is also known with the kernel in the form of an infinite series [5]:

$$u(x, y) = \int\limits_{\mathbb{R}^2} \varphi(t) \left[\frac{\partial}{\partial \tau} G(x, y, t, \tau) \right]_{\tau=0} dt + \int\limits_{\mathbb{R}^2} \psi(t) G(x, y, t, a) \, dt,$$

where

$$G\left(x, y, t, \tau\right) = \frac{1}{4\pi} \sum_{n=-\infty}^{\infty} \left(\frac{1}{r_{n1}} - \frac{1}{r_{n2}}\right);$$

$$r_{n1} = \sqrt{|x - t|^2 + [y - (-1)^n \tau - 2na]^2};$$

$$r_{n2} = \sqrt{|x - t|^2 + [y + (-1)^n \tau - 2na]^2}.$$

Solution to the mixed boundary value problem for an infinite layer in four-dimensional space. Let us show how to apply the recurrence relation for the solution to the mixed boundary value problem for spaces of arbitrary dimension illustrated by the example of an infinite layer in the four-dimensional space. We find kernels from the recurrent formulae.

$$K_3(x, y) = -\frac{1}{2\pi r}\frac{\partial}{\partial r}K_1(r, y) = -\frac{1}{2\pi r}\frac{\partial}{\partial r}\left(\frac{1}{a}\frac{\sin\left(\frac{\pi y}{2a}\right)\operatorname{ch}\left(\frac{\pi r}{2a}\right)}{\operatorname{ch}\left(\frac{\pi r}{a}\right) - \cos\left(\frac{\pi y}{a}\right)}\right) =$$

$$= \frac{\left(2 + \cos\left(\frac{\pi y}{a}\right) + \operatorname{ch}\left(\frac{\pi r}{a}\right)\right)\sin\left(\frac{\pi y}{2a}\right)\operatorname{sh}\left(\frac{\pi r}{2a}\right)}{4a^2 r\left(\cos\left(\frac{\pi y}{a}\right) - \operatorname{ch}\left(\frac{\pi r}{a}\right)\right)^2} =$$

$$= \frac{\left(2 + \cos\left(\frac{\pi y}{a}\right) + \operatorname{ch}\left(\frac{\pi\sqrt{x_1^2 + x_2^2 + x_3^2}}{a}\right)\right)\sin\left(\frac{\pi y}{2a}\right)\operatorname{sh}\left(\frac{\pi\sqrt{x_1^2 + x_2^2 + x_3^2}}{2a}\right)}{4a^2\sqrt{x_1^2 + x_2^2 + x_3^2}\left(\cos\left(\frac{\pi y}{a}\right) - \operatorname{ch}\left(\frac{\pi\sqrt{x_1^2 + x_2^2 + x_3^2}}{a}\right)\right)^2};$$

$$L_3\left(x, y\right) = -\frac{1}{2\pi r}\frac{\partial}{\partial r}L_1\left(r, y\right) = -\frac{1}{2\pi r}\frac{\partial}{\partial r}\left(\frac{1}{2\pi}\ln\left(\frac{\operatorname{ch}\left(\frac{\pi r}{2a}\right) + \sin\left(\frac{\pi y}{2a}\right)}{\operatorname{ch}\left(\frac{\pi r}{2a}\right) - \sin\left(\frac{\pi y}{2a}\right)}\right)\right) =$$

$$= \frac{\sin\left(\frac{\pi y}{2a}\right)\operatorname{sh}\left(\frac{\pi r}{2a}\right)}{2\pi a r\left(\cos\left(\frac{\pi y}{a}\right) + \operatorname{ch}\left(\frac{\pi r}{a}\right)\right)} =$$

$$= \frac{\sin\left(\frac{\pi y}{2a}\right)\operatorname{sh}\left(\frac{\pi\sqrt{x_1^2 + x_2^2 + x_3^2}}{2a}\right)}{2\pi a\sqrt{x_1^2 + x_2^2 + x_3^2}\left(\cos\left(\frac{\pi y}{a}\right) + \operatorname{ch}\left(\frac{\pi\sqrt{x_1^2 + x_2^2 + x_3^2}}{a}\right)\right)}.$$

If the functions $\varphi\left(x\right)$ and $\psi\left(x\right)$ are normal functions of polynomial growth, the solution to the mixed problem is written as the integral formula

$$u(x,y) = \frac{\sin\left(\frac{\pi y}{2a}\right)}{4a^2} \int_{\mathbb{R}^3} \varphi(t) \frac{\left(2 + \cos\left(\frac{\pi y}{a}\right) + \operatorname{ch}\left(\frac{\pi|x-t|}{a}\right)\right) \operatorname{sh}\left(\frac{\pi|x-t|}{2a}\right)}{|x-t| \left(\cos\left(\frac{\pi y}{a}\right) - \operatorname{ch}\left(\frac{\pi|x-t|}{a}\right)\right)^2} dt +$$

$$+ \frac{\sin\left(\frac{\pi y}{2a}\right)}{2\pi a} \int_{\mathbb{R}^3} \psi(t) \frac{\operatorname{sh}\left(\frac{\pi|x-t|}{2a}\right)}{|x-t| \left(\cos\left(\frac{\pi y}{a}\right) + \operatorname{ch}\left(\frac{\pi|x-t|}{a}\right)\right)} dt.$$

Application in the underground hydrodynamics. We consider the solution to the problem of filtration theory describing the flow under the spot dam with an aquitard as an example of the formula for the two variables application. Filtering the liquid (water) is caused by the pressure difference at the upstream wall ($P_1 = -\varphi_1$) and downstream wall ($P_2 = -\varphi_2$) (Figure). The velocity field of the filtered fluid is described by the vector $\overrightarrow{v} = k \overrightarrow{\operatorname{grad} u}$, where the coefficient k characterizes the permeability of the medium (soil) [10, 11]:

$$\Delta u(x,y) = 0, \quad -\infty < x < \infty, \quad 0 < y < a;$$

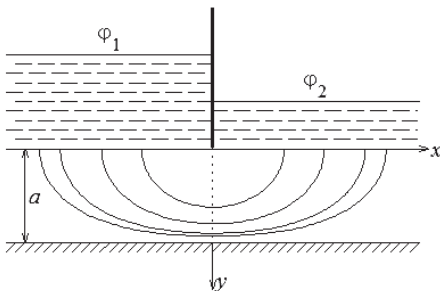$$u(x,0) = \varphi_1, \quad x < 0, \quad u(x,0) = \varphi_2, \quad x > 0;$$

$$u_y(x,a) = 0, \quad -\infty < x < \infty.$$

The solution to this problem is a function

$$u(x,y) = \frac{\varphi_1}{a} \sin\left(\frac{\pi y}{2a}\right) \int_{-\infty}^{0} \frac{\operatorname{ch}(\pi(x-t)/2a)}{\operatorname{ch}(\pi(x-t)/a) - \cos(\pi y/a)} dt +$$

$$+ \frac{\varphi_2}{a} \sin\left(\frac{\pi y}{2a}\right) \int_{0}^{\infty} \frac{\operatorname{ch}(\pi(x-t)/2a)}{\operatorname{ch}(\pi(x-t)/a) - \cos(\pi y/a)} dt =$$

$$= \frac{\varphi_2 - \varphi_1}{\pi} \operatorname{arctg}\left(\frac{\operatorname{sh}(\pi x/2a)}{\sin(\pi y/2a)}\right) + \frac{\varphi_1 + \varphi_2}{2}.$$

Filtration flow diagram for the point dam with an aquitard

Conjugate harmonic function has the form of an equation of the streamlines (see. figure).

$$v\left(x, y\right) = \frac{\varphi_1 - \varphi_2}{2\pi} \ln\left(\frac{\operatorname{ch}\left(\pi x/2a\right) + \cos\left(\pi y/2a\right)}{\operatorname{ch}\left(\pi x/2a\right) - \cos\left(\pi y/2a\right)}\right),$$

Conclusion. The obtained formulae for the solution to a mixed boundary value problem for the Laplace equation in an infinite layer can also be applied when the boundary values are the tempered distributions. In this case the solution is a generalized one, i.e. $u\left(x, y\right)$ and $u_y\left(x, y\right)$ when $y \rightarrow +0$ and thus $y \rightarrow a - 0$ converge weakly in the space $\mathcal{S}'(\mathbb{R}^n)$ to the given boundary values. In case of two variables the specified formulae can be used to solve plane problems of the underground fluid dynamics (the theory of filtration).

REFERENCES

[1] Komech A.I. Linear partial differential equations with constant coefficients. *Itogi Nauki i Tekhniki. Ser. Sovrem. Probl. Mat.: Fund. Napr.* [Results of science and technology. Ser. "Modern Problems of Mathematics: Fundamental Directions"], 1988, vol. 31, pp. 127–261 (in Russ.). Available at: http://www.mathnet.ru/php/archive.phtml?wshow=paper&jrnid=intf&paperid=113&option_lang=eng (accessed 01.11.2014).

[2] Mikhlin S.G. Lineynye uravneniya v chastnykh proizvodnykh [Linear partial differential equations]. Moscow, Vysshaya Shkola Publ., 1977. 430 p.

[3] Bitsadze A.V. Uravneniya matematicheskoy fiziki [Equations of mathematical physics]. Moscow, Fizmatgiz Publ., 1976. 296 p. (Engl. ed.: Bitsadze A.V. Equations of mathematical physics. Moscow, Mir Publishers, 1980. 318 p.). Available at: http://www.amazon.com/Equations-Mathematical-Physics-A-V-Bitsadze/dp/0714715433 (accessed 01.11.2014).

[4] Tikhonov A.N., Samarskiy A.A. Uravneniya matematicheskoy fiziki [Equations of mathematical physics]. Moscow, MGU Publ., 1999. 798 p. (Engl. ed.: Tikhonov A.N., Samarskii A.A. Equations of mathematical physics. Dover Publications; Reprint edition, 2011. 800 p.). Available at: http://www.amazon.com/Equations-Mathematical-Physics-Dover-Books/dp/0486664228 (accessed 01.11.2014).

[5] Polyanin A.D. Spravochnik po lineynym uravneniyam matematicheskoy fiziki [Handbook of linear equations of mathematical physics]. Moscow, Fizmatlit Publ., 2001. 576 p. (Engl. ed.: Polyanin A.D. Handbook of linear partial differential equations for engineers and scientists. USA, Chapman and Hall/CRC, 2001. 800 p.).

[6] Kas'yanov E.Yu., Kopaev A.V. On the solution of the Dirichlet problem for some multidimensional domains by the method of reproducing kernels. *Izv. Vyssh. Uchebn. Zaved., Mat.* [Russ. Math.], 1991, no. 6, pp. 17–20 (in Russ.).

[7] Vladimirov V.S. Obobshchennye funktsii v matematicheskoy fizike [Tempered distributions in mathematical physics]. Moscow, Nauka Publ., 1979. 320 p.

[8] Bochner S. Vorlesungen ьber Fouriersche Integrale. Leipzig, Akademische Verlagsgesellschaft m.b. H. VIII, 1932. 229 p. (Engl. ed.: Bochner S. Lectures on Fourier integrals. Trans. from the Ger. Princeton Uni. Press, 1959. 333 p. Russ. ed.: Bokhner S. Lektsii ob integralakh Fur'e [Lectures on Fourier integrals]. Moscow, Fizmatgiz Publ., 1962. 360 p.).

[9] Ditkin V.A., Prudnikov A.P. Integral'nye preobrazovaniya i operatsionnoe ischislenie [Integral transforms and operator calculus]. Moscow, Fizmatgiz Publ., 1961. 524 p.

[10] Radygin V.M., Golubeva O.V. Primenenie funktsiy kompleksnogo peremennogo v zadachakh fiziki i tekhniki [Application of function of complex variable in physics and engineering problems]. Moscow, Vysshaya Shkola Publ., 1983. 160 p.

[11] Golubeva O.V. Kurs mekhaniki sploshnykh sred [The course of continuum mechanics]. Moscow, Vysshaya Shkola Publ., 1972. 367 p.

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