ДИФФЕРЕНЦИАЛЬНЫЕ УРАВНЕНИЯ, ДИНАМИЧЕСКИЕ СИСТЕМЫ И ОПТИМАЛЬНОЕ УПРАВЛЕНИЕ

УДК 517.91

ОБ АСИМПТОТИЧЕСКОЙ КЛАССИФИКАЦИИ РЕШЕНИЙ НЕЛИНЕЙНЫХ УРАВНЕНИЙ ТРЕТЬЕГО И ЧЕТВЕРТОГО ПОРЯДКОВ СО СТЕПЕННОЙ НЕЛИНЕЙНОСТЬЮ

И.В. Асташова

Московский государственный университет им. М.В. Ломоносова,
Москва, Российская Федерация
Московский государственный университет экономики, статистики
и информатики (МЭСИ), Москва, Российская Федерация
e-mail: ast@diffiety.ac.ru

Исследовано асимптотическое поведение всех решений нелинейных дифференциальных уравнений типа Эмдена – Фаулера третьего и четвертого порядков. Приведены ранее полученные автором настоящей статьи результаты. Уравнение n-го порядка сведено к системе на $(n-1)$-мерной сфере. С помощью исследования асимптотического поведения всех возможных траекторий системы получена асимптотическая классификация решений исходного уравнения.

***Ключевые слова*:** нелинейное дифференциальное уравнение высокого порядка, асимптотическое поведение решений, качественные свойства, асимптотическая классификация решений.

ON ASYMPTOTIC CLASSIFICATION OF SOLUTIONS TO NONLINEAR THIRD- AND FOURTH-ORDER DIFFERENTIAL EQUATIONS WITH POWER NONLINEARITY

I.V. Astashova

Lomonosov Moscow State University, Moscow, Russian Federation
Moscow State University of Economics, Statistics and Informatics (MESI),
Moscow, Russian Federation
e-mail: ast@diffiety.ac.ru

The asymptotic behavior of all solutions to the fourth and the third order Emden – Fowler type differential equation is investigated. The author's previously obtained results are supplemented. The equation of the n-th order is transformed into a system on the $(n-1)$-dimensional sphere. By the investigation of asymptotic behavior to all possible trajectories of this system the asymptotic classification of all solutions to the equation is obtained.

***Keywords*:** nonlinear higher-order ordinary differential equation, asymptotic behavior, qualitative properties, asymptotic classification of solutions.

Introduction. The investigation of asymptotic behavior of solutions to nonlinear differential equations near the boundaries of their domain and the classification of all possible solutions to this equations is one of the major problems in qualitative theory of differential equations. This problem is one of the most important because there are no general methods for investigation of qualitative properties of solutions to nonlinear differential equations. Note that Emden – Fowler equation appears for the first time in [1]. Its physical origin is also described in [2]. This equation was investigated in detail in the books [3, 4], and later in [5]. See also [6, 7] and references. Asymptotic properties of solutions to different generalizations of this equation were investigated in [8–35]. The results concerning asymptotic behavior of solutions to nonlinear ordinary differential equations is used to describe the properties of solutions to nonlinear partial differential equations. See, for example, [36–40].

In this article the asymptotic classification of all possible solutions to the fourth order Emden – Fowler type differential equations

$$y^{\mathrm{IV}}(x) + p_0\,|y|^{k-1}\,y(x) = 0, \quad k > 1, \; p_0 > 0 \tag{1}$$

and

$$y^{\mathrm{IV}}(x) - p_0\,|y|^{k-1}\,y(x) = 0, \quad k > 1, \; p_0 > 0 \tag{2}$$

is given.

The asymptotic classification of all possible solutions to the third order Emden – Fowler type differential equations

$$y^{\mathrm{III}}(x) + p(x)\,|y|^{k-1}\,y(x) = 0, \quad k > 1, \; p(x) > 0 \tag{3}$$

is described.

For fourth-order nonlinear equations, the oscillatory problem was investigated in [10, 13, 14, 17, 21, 28, 29, 31, 33, 35], in linear case — in [41].

Phase Sphere. Note that if a function $y(x)$ is a solution to equation (1), the same is true for the function

$$z(x) = Ay(Bx + C), \tag{4}$$

where $A \neq 0$, $B > 0$, and C are any constants satisfying

$$|A|^{k-1} = B^4. \tag{5}$$

Indeed, we have

$$\begin{aligned} z^{\mathrm{IV}}(x) + p_0\,|z|^{k-1}\,z(x) &= AB^4 y^{\mathrm{IV}}(Bx + C) + \\ &+ p_0\,|Ay(Bx + C)|^{k-1}\,Ay(Bx + C) = \\ &= Ay^{\mathrm{IV}}(Bx + C)\left(B^4 - |A|^{k-1}\right) = 0. \end{aligned}$$

Any non-trivial solution $y(x)$ to equation (1) generates a curve $(y(x), y'(x), y''(x), y'''(x))$ in $\mathbb{R}^4\backslash\{0\}$. Let us introduce in $\mathbb{R}^4\backslash\{0\}$ an equivalence relation such that two solutions connected by (4), (5) generate equivalent curves, i.e. the curves passing through equivalent points (may be for different x).

We assume that points (y_0, y_1, y_2, y_3) and (z_0, z_1, z_2, z_3) in $\mathbb{R}^4\backslash\{0\}$ are equivalent if there exists a positive constant λ such that $z_j = \lambda^{4+j(k-1)} y_j$, $j = 0, 1, 2, 3$.

The factor space obtained is homeomorphic to the three-dimensional sphere $S^3 = \{y \in \mathbb{R}^4 : y_0^2 + y_1^2 + y_2^2 + y_3^2 = 1\}$. On this sphere there is exactly one representative of each equivalence class because for any point $(y_0, y_1, y_2, y_3) \in \mathbb{R}^4\backslash\{0\}$ the equation $\lambda^8 y_0^2 + \lambda^{2k+6} y_1^2 + \lambda^{4k+4} y_2^2 + \lambda^{6k+2} y_3^2 = 1$ has exactly one positive root λ.

It is possible to construct another hyper-surface in $\mathbb{R}^4$ with a single representative of each equivalence class, namely,

$$E = \left\{ y \in \mathbb{R}^4 : \sum_{j=0}^{3} |y_j|^{\frac{1}{j(k-1)+4}} = 1 \right\}. \tag{6}$$

We define $\Phi_S : \mathbb{R}^4\backslash\{0\} \to S^3$ and $\Phi_E : \mathbb{R}^4\backslash\{0\} \to E$ as mappings taking each point in $\mathbb{R}^4\backslash\{0\}$ to the equivalent point in S^3 or E. Note that the restrictions $\Phi_S|_E$ and $\Phi_E|_{S^3}$ are inverse homeomorphisms.

Lemma 1. *There is a dynamical system on the sphere S^3 such that all its trajectories can be obtained by the mapping Φ_S from the curves generated in $\mathbb{R}^4\backslash\{0\}$ by nontrivial solutions to equation* (1)*. Conversely, any nontrivial solution to equation* (1) *generates in $\mathbb{R}^4\backslash\{0\}$ a curve whose image under Φ_S is a trajectory of the above dynamical system.*

◀ First we define on the sphere S^3 a smooth structure using an atlas consisting of eight charts.

The two semi-spheres defined by the inequalities $y_0 > 0$ and $y_0 < 0$ are covered by the charts with the coordinate functions (respectively u_1^+, u_2^+, u_3^+ and u_1^-, u_2^-, u_3^-) defined by the formulae $u_j^\pm = y_j\,|y_0|^{-\frac{4+j(k-1)}{4}} \operatorname{sgn} y_0$, $j = 1, 2, 3$.

The semi-spheres defined by the inequalities $y_1 > 0$ and $y_1 < 0$ are covered by the charts with the coordinate functions (respectively v_0^+, v_2^+, v_3^+ and v_0^-, v_2^-, v_3^-) defined as

$$v_j^\pm = y_j\,|y_1|^{-\frac{4+j(k-1)}{k+3}} \operatorname{sgn} y_1, \qquad j = 0, 2, 3.$$

The semi-spheres defined by the inequalities $y_2 > 0$ and $y_2 < 0$ are covered by the charts with the coordinate functions (respectively w_0^+, w_1^+, w_3^+ and w_0^-, w_1^-, w_3^-) defined as $w_j^\pm = y_j\,|y_2|^{-\frac{4+j(k-1)}{2k+2}} \operatorname{sgn} y_2$, $j = 0, 1, 3$.

Finally, the semi-spheres defined by the inequalities $y_3 > 0$ and $y_3 < 0$ are covered by the charts with the coordinate functions (respectively g_0^+, g_1^+, g_2^+ and g_0^-, g_1^-, g_2^-) defined as $g_j^\pm = y_j\,|y_3|^{-\frac{4+j(k-1)}{3k+1}} \operatorname{sgn} y_3$, $j = 0, 1, 2$.

Note that each of these coordinate functions can be defined by its own formula on the whole corresponding semi-space ($y_j \gtrless 0$) and it takes equivalent points to the same value. This fact facilitates description of the trajectories generated on S^3 by solutions to equation (1). To be more precise, by their restrictions on the intervals where some derivative has constant sign.

E.g., when a solution is positive, the trajectory generated can be described by the following differential equations:

$$\frac{du_1^+}{dx} = y'' \left|y\right|^{-\frac{k+3}{4}} \operatorname{sgn} y - \frac{k+3}{4} y'^2 \left|y\right|^{-\frac{k+7}{4}} = \\ = \left|y\right|^{\frac{k-1}{4}} \left(u_2^+ - \frac{k+3}{4} u_1^{+2} \right);$$

$$\frac{du_2^+}{dx} = y''' \left|y\right|^{-\frac{2k+2}{4}} \operatorname{sgn} y - \frac{2k+2}{4} y' y'' \left|y\right|^{-\frac{2k+6}{4}} = \\ = \left|y\right|^{\frac{k-1}{4}} \left(u_3^+ - \frac{2k+2}{4} u_1^+ u_2^+ \right);$$

$$\frac{du_3^+}{dx} = -p_0 \left|y\right|^{k-\frac{3k+1}{4}} - \frac{3k+1}{4} y' y''' \left|y\right|^{-\frac{3k+5}{4}} = \\ = \left|y\right|^{\frac{k-1}{4}} \left(-p_0 - \frac{3k+1}{4} u_1^+ u_3^+ \right).$$

Parameterizing it by $t_u = \int\limits_{x_0}^{x} \left|y\right|^{\frac{k-1}{4}} dx$, we obtain its internal description in terms of u_j^+ :

$$\frac{du_1^+}{dt_u} = u_2^+ - \frac{k+3}{4} u_1^{+2};$$

$$\frac{du_2^+}{dt_u} = u_3^+ - \frac{2k+2}{4} u_1^+ u_2^+;$$

$$\frac{du_3^+}{dt_u} = -p_0 - \frac{3k+1}{4} u_1^+ u_3^+.$$

The same equations appear for (u_1^-, u_2^-, u_3^-). Similar calculations yield equations for other charts:

$$\frac{dv_0^\pm}{dt_v} = 1 - \frac{4}{k+3} v_0^\pm v_2^\pm; \qquad \frac{dw_0^\pm}{dt_w} = w_1^\pm - \frac{4}{2k+2} w_0^\pm w_3^\pm;$$

$$\frac{dv_2^\pm}{dt_v} = v_3^\pm - \frac{2k+2}{k+3} v_2^{\pm 2}; \qquad \frac{dw_1^\pm}{dt_w} = 1 - \frac{k+3}{2k+2} w_1^\pm w_3^\pm;$$

$$\frac{dv_3^\pm}{dt_v} = -p_0 \left|v_0^\pm\right|^k \operatorname{sgn} v_0^\pm - \frac{3k+1}{k+3} v_2^\pm v_3^\pm, \qquad \frac{dw_3^\pm}{dt_w} = -p_0 \left|w_0^\pm\right|^k \operatorname{sgn} w_0^\pm - \frac{3k+1}{2k+2} w_3^{\pm 2},$$

$$
\begin{aligned}
\frac{dg_0^{\pm}}{dt_q} &= g_1^{\pm} + \frac{4}{3k+1} p_0 \left| g_0^{\pm} \right|^{k+1};\\
\frac{dg_1^{\pm}}{dt_q} &= g_2^{\pm} + \frac{k+3}{3k+1} p_0 g_1^{\pm} \left| g_0^{\pm} \right|^{k} \operatorname{sgn} g_0^{\pm};\\
\frac{dg_2^{\pm}}{dt_q} &= 1 + \frac{2k+2}{3k+1} p_0 g_2^{\pm} \left| g_0^{\pm} \right|^{k} \operatorname{sgn} g_0^{\pm}.
\end{aligned}
$$

Using a partition of unity one can obtain a dynamical system on the whole sphere S^3 to describe all trajectories generated by nontrivial solutions to equation (1). ►

Typical and Non-Typical Solutions. Now we consider the space $\mathbb{R}^4$ as the union of its $16 = 2^4$ closed subsets defined according to different combinations of signs of the four coordinates. Denote these sets by $\begin{bmatrix} \pm \\ \pm \\ \pm \\ \pm \end{bmatrix} \subset \mathbb{R}^4$, where each sign $\pm$ can be substituted by $+$, or $-$, or 0 (for boundary points). For example,

$$
\begin{bmatrix} + \\ + \\ 0 \\ - \end{bmatrix} = \left\{ y \in \mathbb{R}^4 : y_0 \geq 0, y_1 \geq 0, y_2 = 0, y_3 \leq 0, \right\}.
$$

Besides, let Ω_- and Ω_+ denote respectively

$$
\begin{bmatrix} + \\ - \\ + \\ - \end{bmatrix} \cup \begin{bmatrix} + \\ - \\ + \\ + \end{bmatrix} \cup \begin{bmatrix} + \\ - \\ - \\ + \end{bmatrix} \cup \begin{bmatrix} + \\ + \\ - \\ + \end{bmatrix} \cup \begin{bmatrix} - \\ + \\ - \\ + \end{bmatrix} \cup \begin{bmatrix} - \\ + \\ - \\ - \end{bmatrix} \cup \begin{bmatrix} - \\ + \\ + \\ - \end{bmatrix} \cup \begin{bmatrix} - \\ - \\ + \\ - \end{bmatrix}
$$

and

$$
\begin{bmatrix} + \\ + \\ + \\ + \end{bmatrix} \cup \begin{bmatrix} + \\ + \\ + \\ - \end{bmatrix} \cup \begin{bmatrix} + \\ + \\ - \\ - \end{bmatrix} \cup \begin{bmatrix} + \\ - \\ - \\ - \end{bmatrix} \cup \begin{bmatrix} - \\ - \\ - \\ - \end{bmatrix} \cup \begin{bmatrix} - \\ - \\ - \\ + \end{bmatrix} \cup \begin{bmatrix} - \\ - \\ + \\ + \end{bmatrix} \cup \begin{bmatrix} - \\ + \\ + \\ + \end{bmatrix}.
$$

Note, that the sets Ω_- and Ω_+ cover the whole space $\mathbb{R}^4$, intersect only along their common boundary, and can be obtained from each other using the mapping $(y_0, y_1, y_2, y_3) \in \mathbb{R}^4 \to (y_0, -y_1, y_2, -y_3) \in \mathbb{R}^4$, which corresponds to changing the sign of the independent variable ($x \to -x$).

Lemma 2. *The sets $\Omega_- \cap S^3$, $\Omega_+ \cap S^3$, $\Omega_- \cap E$, and $\Omega_+ \cap E$ are homeomorphic to the solid torus.*

◀ It is sufficient to consider $\Omega_+ \cap S^3$. The set Ω_+ is the union of its two homeomorphic subsets

$$\Omega_{++} = \begin{bmatrix} + \\ + \\ + \\ + \end{bmatrix} \cup \begin{bmatrix} + \\ + \\ + \\ - \end{bmatrix} \cup \begin{bmatrix} + \\ + \\ - \\ - \end{bmatrix} \cup \begin{bmatrix} + \\ - \\ - \\ - \end{bmatrix}$$

and

$$\Omega_{+-} = \begin{bmatrix} - \\ - \\ - \\ - \end{bmatrix} \cup \begin{bmatrix} - \\ - \\ - \\ + \end{bmatrix} \cup \begin{bmatrix} - \\ - \\ + \\ + \end{bmatrix} \cup \begin{bmatrix} - \\ + \\ + \\ + \end{bmatrix}.$$

In order to describe the set $\Omega_{++} \cap S^3$, we use the stereographic projection $S^3 \backslash \{(-1,0,0,0)\} \to \mathbb{R}^3$ (Fig. 1).

The image of $\Omega_{++} \cap S^3$ under this projection is contained in the ball of radius 2 and is equal to the union of its two quarters, which is homeomorphic to the 3-dimensional ball. The same is true for $\Omega_{+-} \cap S^3$.

The intersection $(\Omega_{++} \cap S^3) \cap (\Omega_{+-} \cap S^3) = \left(\begin{bmatrix} 0 \\ + \\ + \\ + \end{bmatrix} \cup \begin{bmatrix} 0 \\ - \\ - \\ - \end{bmatrix} \right) \cap S^3$

maps to the disjoint union of two spherical triangles (2-dimensional figures,

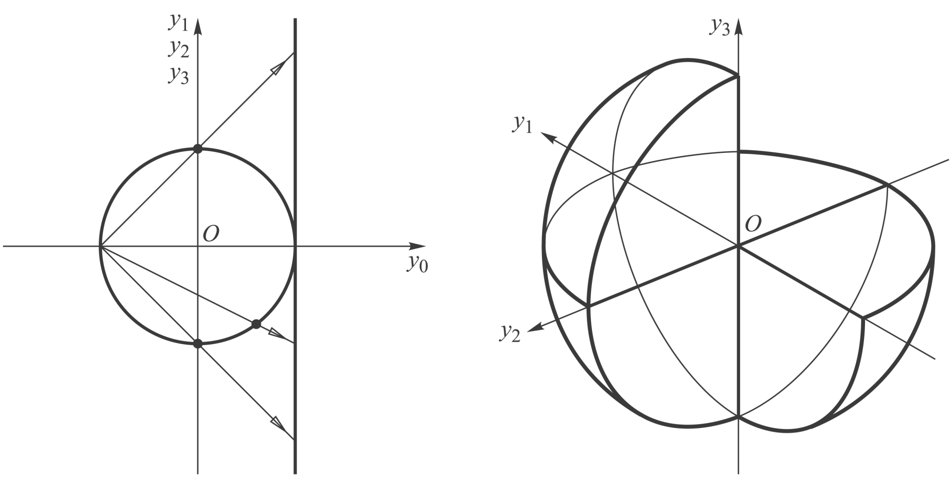

Fig. 1. Stereographic projection and its image of $\Omega_{++} \cap S^3$

not their boundaries). Thus, the set $\Omega_+ \cap S^3$ is homeomorphic to the pair of two balls glued along two disjoint triangles, which is equivalent to the solid torus. ►

Lemma 3. *Any trajectory in $\mathbb{R}^4$ generated by a non-trivial solution to* (1) *either completely lies inside one of the sets Ω_- and Ω_+* (*i.e., in their interior*)*, or consists of two parts, first inside Ω_- and another inside Ω_+ with a single point in their common boundary.*

◄ For the trajectories generated by solutions to equation (1), consider all possible passages between the sets $\begin{bmatrix} \pm \\ \pm \\ \pm \\ \pm \end{bmatrix}$.

Inside Ω_+ the only possible passages are

$$\begin{array}{ccccccc}
\begin{bmatrix} + \\ + \\ + \\ + \end{bmatrix} & \rightarrow & \begin{bmatrix} + \\ + \\ + \\ - \end{bmatrix} & \rightarrow & \begin{bmatrix} + \\ + \\ - \\ - \end{bmatrix} & \rightarrow & \begin{bmatrix} + \\ - \\ - \\ - \end{bmatrix} \\
\uparrow & & & & & & \downarrow \\
\begin{bmatrix} - \\ + \\ + \\ + \end{bmatrix} & \leftarrow & \begin{bmatrix} - \\ - \\ + \\ + \end{bmatrix} & \leftarrow & \begin{bmatrix} - \\ - \\ - \\ + \end{bmatrix} & \leftarrow & \begin{bmatrix} - \\ - \\ - \\ - \end{bmatrix}
\end{array}, \tag{7}$$

inside Ω_- they are

$$\begin{array}{ccccccc}
\begin{bmatrix} + \\ - \\ + \\ - \end{bmatrix} & \leftarrow & \begin{bmatrix} + \\ - \\ + \\ + \end{bmatrix} & \leftarrow & \begin{bmatrix} + \\ - \\ - \\ + \end{bmatrix} & \leftarrow & \begin{bmatrix} + \\ + \\ - \\ + \end{bmatrix} \\
\downarrow & & & & & & \uparrow \\
\begin{bmatrix} - \\ - \\ + \\ - \end{bmatrix} & \rightarrow & \begin{bmatrix} - \\ + \\ + \\ - \end{bmatrix} & \rightarrow & \begin{bmatrix} - \\ + \\ - \\ - \end{bmatrix} & \rightarrow & \begin{bmatrix} - \\ + \\ - \\ + \end{bmatrix}
\end{array}, \tag{8}$$

and the only possible passages between Ω_- and Ω_+ are

$$\begin{bmatrix} + \\ + \\ + \\ - \end{bmatrix} \leftarrow \begin{bmatrix} + \\ - \\ + \\ - \end{bmatrix} \rightarrow \begin{bmatrix} + \\ - \\ - \\ - \end{bmatrix}, \qquad \begin{bmatrix} - \\ - \\ + \\ + \end{bmatrix} \leftarrow \begin{bmatrix} + \\ - \\ + \\ + \end{bmatrix} \rightarrow \begin{bmatrix} + \\ + \\ + \\ + \end{bmatrix},$$

$$\begin{bmatrix} - \\ - \\ - \\ + \end{bmatrix} \leftarrow \begin{bmatrix} + \\ - \\ - \\ + \end{bmatrix} \rightarrow \begin{bmatrix} + \\ - \\ - \\ - \end{bmatrix}, \qquad \begin{bmatrix} + \\ + \\ + \\ + \end{bmatrix} \leftarrow \begin{bmatrix} + \\ + \\ - \\ + \end{bmatrix} \rightarrow \begin{bmatrix} + \\ + \\ - \\ - \end{bmatrix},$$

$$\begin{bmatrix} - \\ - \\ - \\ + \end{bmatrix} \leftarrow \begin{bmatrix} - \\ + \\ - \\ + \end{bmatrix} \rightarrow \begin{bmatrix} - \\ + \\ + \\ + \end{bmatrix}, \qquad \begin{bmatrix} + \\ + \\ - \\ - \end{bmatrix} \leftarrow \begin{bmatrix} - \\ + \\ - \\ - \end{bmatrix} \rightarrow \begin{bmatrix} - \\ - \\ - \\ - \end{bmatrix},$$

$$\begin{bmatrix} + \\ + \\ + \\ - \end{bmatrix} \leftarrow \begin{bmatrix} - \\ + \\ + \\ - \end{bmatrix} \rightarrow \begin{bmatrix} - \\ + \\ + \\ + \end{bmatrix}, \qquad \begin{bmatrix} - \\ - \\ - \\ - \end{bmatrix} \leftarrow \begin{bmatrix} - \\ - \\ + \\ - \end{bmatrix} \rightarrow \begin{bmatrix} - \\ - \\ + \\ + \end{bmatrix},$$

always from Ω_- to Ω_+.

So, any trajectory generated by a non-trivial solution can perform only one passage between Ω_- and Ω_+, which can be only from Ω_- to Ω_+. ►

Lemma 4. *There exist trajectories of all three types mentioned in Lemma 3, namely*

- *trajectories lying completely in* Ω_-;
- *trajectories lying completely in* Ω_+;
- *trajectories with a single passage* $\Omega_- \to \Omega_+$.

◄ Any solution to (1) with initial data corresponding to a point from $\Omega_- \cap \Omega_+$ generates a trajectory of the 3rd type. E.g., the solution with initial data $y'(0) = 0$, $y(0) = y''(0) = y'''(0) = 1$ generates a trajectory with the passage

$$\begin{bmatrix} + \\ - \\ + \\ + \end{bmatrix} \subset \Omega_- \to \begin{bmatrix} + \\ + \\ + \\ + \end{bmatrix} \subset \Omega_+.$$

If there exists a solution $y(x)$ to (1) generating a trajectory lying completely in Ω_-, then the function $z(x) = y(-x)$ is also a solution to (1) and generates a trajectory completely lying in Ω_+. So, we have to prove existence of a trajectory of the first type.

Assume the converse. Then any trajectory passing through a point $s \in \Omega_- \cap S^3$ must reach the boundary $\partial\Omega_- \cap S^3$. Thus we obtain the mapping $\Omega_- \cap S^3 \to \partial\Omega_- \cap S^3$.

To prove its continuity we represent it as $s \in \Omega_- \cap S^3 \mapsto \mathrm{Traj}_{p_0}(s, \xi(s)) \in$ $\in \partial\Omega_- \cap S^3$.

Here $\mathrm{Traj}_{p_0}(s,t)$ is the point in S^3 reached at the time t by the trajectory of the dynamical system on the sphere that passed s at the time 0. The mapping $\mathrm{Traj}_{p_0} : S^3 \times \mathbb{R} \to S^3$ is continuous according to the general properties of differential equations.

The function $\xi : \Omega_- \cap S^3 \to \mathbb{R}$ gives the time t at which the trajectory passing through the given point of Ω_- at $t_0 = 0$ reaches $\partial\Omega_-$. Now we prove continuity of ξ.

Suppose $\xi(s_1) = t_1$ and $\varepsilon > 0$. Then, since $\mathrm{Traj}_{p_0}(s_1, t_1 + \varepsilon)$ is inside Ω_+, there exists a neighborhood U_+ of s_1 such that for any $s \in U_+$ the point $\mathrm{Traj}_{p_0}(s, t_1 + \varepsilon)$ is also inside Ω_+. So, we have $\xi(s) < t_1 + \varepsilon$ for all $s \in U_+$.

Similarly, since $\mathrm{Traj}_{p_0}(s_1, t_1 - \varepsilon)$ is inside Ω_-, there exists a neighborhood U_- of s_1 such that for any $s \in U_-$ the point $\mathrm{Traj}_{p_0}(s, t_1 - \varepsilon)$ is also inside Ω_-, whence $\xi(s) > t_1 - \varepsilon$.

So, for all $s \in U_- \cap U_+$ we have $|\xi(s) - t_1| < \varepsilon$. Thus $\xi(s)$ is continuous on $\Omega_- \cap S^3$ and we have the continuous mapping $\Omega_- \cap S^3 \to \partial\Omega_- \cap S^3$ whose restriction to $\partial\Omega_- \cap S^3$ is the identity map. In other words, we have the composition $\partial\Omega_- \cap S^3 \to \Omega_- \cap S^3 \to \partial\Omega_- \cap S^3$, which is the identity map, inducing the identity map on the homology groups: $H_2(\partial\Omega_- \cap S^3) \to H_2(\Omega_- \cap S^3) \to H_2(\partial\Omega_- \cap S^3)$.

Since $\Omega_- \cap S^3$ and $\partial\Omega_- \cap S^3$ are homeomorphic to the solid torus and the torus surface respectively, the above composition can be written as $\mathbb{Z} \to 0 \to \mathbb{Z}$, which cannot be the identity mapping. This contradiction proves the lemma. ►

Lemma 5. *Suppose $y(x)$ is a non-trivial solution to equation* (1) *maximally extended to the right. Then neither $y(x)$ nor any of its derivatives $y'(x)$, $y''(x)$, $y'''(x)$ can have constant sign near the right boundary of their domain.*

◄ We prove it for $y(x)$. For the derivatives the proof is just similar.

Suppose $y(x)$ is defined on an interval (x_-, x_+), bounded or not, and is positive in a neighborhood of x_+. Then $y'''(x)$, due to (1), is monotonically decreasing to a finite or infinite limit as $x \to x_+$. Then $y'''(x)$ ultimately has a constant sign. In the same way, $y''(x)$, $y'(x)$, and $y(x)$ itself are all ultimately monotone and have finite or infinite limits as $x \to x_+$.

Suppose $x_+ < +\infty$. If either of the limits mentioned is finite, then all other limits are finite, too, which is impossible for a maximally extended solution. If all limits are infinite, they must have the same sign, which contradicts to equation (1).

Now suppose $x_+ = +\infty$. If either of the limits mentioned is non-zero, then all limits must be infinite and have the same sign, which contradicts to equation (1). If all these limits are zero, then $y(x)$, which is ultimately positive, is decreasing to 0. Hence, $y'(x)$ is ultimately negative and increasing to 0. Similarly, $y''(x)$ is ultimately positive and decreasing to 0, $y'''(x)$ is ultimately negative and increasing to 0, which contradicts to equation (1), since $y(x)$ is ultimately positive. These contradictions prove the lemma. ►

Thus, no trajectory generated in $\mathbb{R}^4$ by a non-trivial solution to (1) can ultimately rest in one of the sets $\begin{bmatrix} \pm \\ \pm \\ \pm \\ \pm \end{bmatrix}$.

Corollary 1. *All maximally extended solutions to equation (1), as well as their derivatives, are oscillatory near both boundaries of their domains.*

Note that according to Lemma 3 we can distinguish two types of asymptotic behavior of oscillatory solutions to equation (1), near the right boundaries of their domains.

Definition 1. *An oscillatory solution to equation* (1) *is called typical* (*to the right*) *if ultimately this solution and its derivatives change their signs according to scheme* (7)*, and non-typical if according to* (8).

Asymptotic Behavior of Typical Solutions. This section is devoted to the asymptotic behavior of typical (to the right) solutions to equation (1), i.e. those generating trajectories ultimately lying inside Ω_+.

Since such a trajectory ultimately admits only the passages shown in (1), there exists an increasing sequence of the points $x_0''' < x_0'' < x_0' < x_0 < \\ < x_1''' < x_1'' < x_1' < x_1 < \ldots$ such that $y(x_j) = y'(x_j') = y''(x_j'') = \\ = y'''(x_j''') = 0$ $(j = 1, 2, \ldots)$, and each point is a zero only for one of the functions $y(x)$, $y'(x)$, $y''(x)$, $y'''(x)$ (Fig. 2). The points x_j, x_j', x_j'', x_j''' will be called the *nodes* of the solution $y(x)$.

For solutions generating trajectories completely lying inside Ω_+, the sequences of their nodes can be indexed by all integers (negative ones, too).

Lemma 6. *Any typical solution $y(x)$ to equation* (1) *satisfies at its nodes the following inequalities:*

$$\left|y(x_j')\right| < \left|y(x_{j+1}''')\right| < \left|y(x_{j+1}'')\right| < \left|y(x_{j+1}')\right|; \tag{9}$$

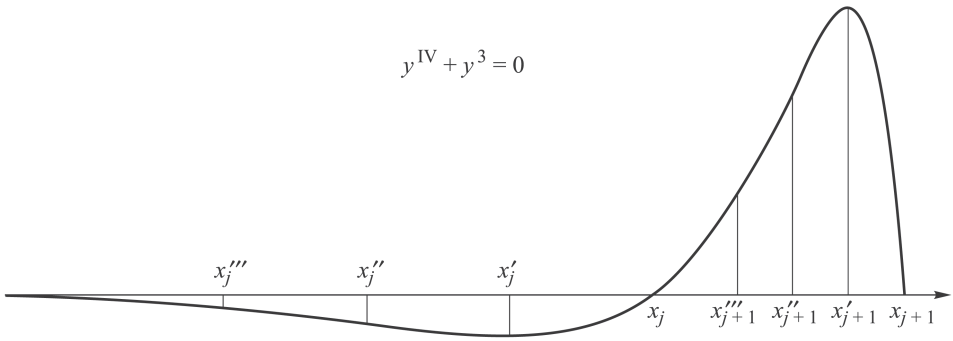

Fig. 2. Zeroes of the derivatives of a typical solution

$$\left|y'(x_j'')\right| < |y'(x_j)| < \left|y'(x_{j+1}''')\right| < \left|y'(x_{j+1}'')\right|; \tag{10}$$

$$\left|y''(x_j''')\right| < \left|y''(x_j')\right| < |y''(x_j)| < \left|y''(x_{j+1}''')\right|; \tag{11}$$

$$|y'''(x_j)| < \left|y'''(x_{j+1}'')\right| < \left|y'''(x_{j+1}')\right| < |y'''(x_{j+1})|. \tag{12}$$

◀ Indeed,

$$\frac{p_0}{k+1}\left(\left|y(x_j')\right|^{k+1} - \left|y(x_{j+1}''')\right|^{k+1}\right) = -\,p_0\int\limits_{x_j'}^{x_{j+1}'''} y'(x)\,|y(x)|^{k-1}\,y(x)\,dx =$$

$$= \int\limits_{x_j'}^{x_{j+1}'''} y'(x)y^{\mathrm{IV}}(x)dx = y'(x)y'''(x)\Big|_{x_j'}^{x_{j+1}'''} - \int\limits_{x_j'}^{x_{j+1}'''} y''(x)y'''(x)\,dx < 0,$$

since $y''(x)y'''(x) > 0$ for all $x \in \left[x_j', x_{j+1}'''\right)$ and $y'(x_j') = y'''(x_{j+1}''') = 0$. This gives the first of inequalities (9), whereas the rest inequalities follow from $y(x)y'(x) > 0$ on the interval $\left[x_{j+1}''', x_{j+1}'\right)$.

Similarly, for the first of inequalities (10) we have $y'(x_j'')^2 - y'(x_j)^2 = = -2\int\limits_{x_j''}^{x_j} y'(x)y''(x)\,dx = -2y(x)y''(x)\Big|_{x_j''}^{x_j} + 2\int\limits_{x_j''}^{x_j} y(x)y'''(x)\,dx < 0$, since $y(x_j) = y''(x_j'') = 0$ and $y(x)y'''(x) < 0$ on $\left[x_j'', x_j\right)$. The rest ones follow from the inequality $y'(x)y''(x) > 0$ on $\left[x_j, x_{j+1}''\right)$.

In the same way, for the first of (11) we have

$$y''(x_j''')^2 - y''(x_j')^2 = -2\int\limits_{x_j'''}^{x_j'} y''(x)y'''(x)\,dx =$$

$$= -2y'(x)y'''(x)\Big|_{x_j'''}^{x_j'} + 2\int\limits_{x_j'''}^{x_j'} y'(x)y^{\mathrm{IV}}(x)\,dx < 0,$$

since $y'(x)y^{\mathrm{IV}}(x) = -p_0 |y|^{k-1} y(x)y'(x) < 0$ on $\left[x_j''', x_j'\right)$ and $y'(x_j') = = y'''(x_j''') = 0$. The rest ones follow from $y''(x)y'''(x) > 0$ on $\left[x_j', x_{j+1}'''\right)$.

Finally, for the first of (12) we have

$$y'''(x_j)^2 - y'''(x_{j+1}'')^2 = -2 \int\limits_{x_j}^{x_{j+1}''} y'''(x)y^{\mathrm{IV}}(x)\,dx =$$

$$= 2p_0 \int\limits_{x_j}^{x_{j+1}''} y'''(x)y(x)\,|y(x)|^{k-1}\,dx = 2p_0 y''(x)y(x)\,|y(x)|^{k-1}\Big|_{x_j}^{x_{j+1}''} -$$

$$-2kp_0 \int_{x_j}^{x_{j+1}''} y''(x)y'(x)\,|y(x)|^{k-1}\,dx < 0,$$

since $y'(x)y''(x) > 0$ on $\left[x_j, x_{j+1}''\right)$ and $y(x_j) = y''(x_{j+1}'') = 0$, whereas the rest inequalities follow from $y'''(x)y^{\mathrm{IV}}(x) > 0$ on $\left[x_{j+1}'', x_{j+1}\right)$. ►

So, the absolute values of the local extrema of any typical solution to equation (1) form a strictly increasing sequence. The same holds for its first, second, and third derivatives.

Hereafter we need some extra notations. Put $\Omega_+^1 = \mathrm{Traj}_1(\Omega_+ \cap S^3, 1) \subset S^3$. This is a compact subset of the interior of Ω_+ containing ultimate parts of all trajectories generated by maximally extended typical solutions to equation (1) with $p_0 = 1$. As for solutions generating the curves in $\mathbb{R}^4$ completely lying in Ω_+, the trajectories related completely lie in Ω_+^1.

Besides, we define the compact sets $K_i = \left\{a \in \Omega_+^1 : a_i = 0\right\}$ and the functions $\xi_j : \mathbb{R}^4 \setminus \{0\} \to \mathbb{R}$, $j = 0, 1, 2, 3$, taking each $a \in \mathbb{R}^4 \setminus \{0\}$ to the minimal positive zero of the derivative $y^{(j)}(x)$ of the solution to the initial data problem

$$\begin{aligned} &y^{\mathrm{IV}}(x) + y(x)\,|y(x)|^{k-1} = 0; \\ &y^{(j)}(0) = a_j, \qquad j = 0, 1, 2, 3. \end{aligned} \tag{13}$$

Further, to each solution $y(x)$ to equation (1) we associate the function $F_y(x) = \sum\limits_{j=0}^{3} \left|\rho y^{(j)}(x)\right|^{\frac{1}{j(k-1)+4}}$ with $\rho = p_0^{\frac{1}{k-1}}$. The notation F_y does not use p_0, since non-trivial functions cannot be solutions to equation (1) with different p_0.

Lemma 7. *The restrictions* $\xi_i|_{K_j}$, $i, j = 0, 1, 2, 3$, *are continuous.*

◄ First we prove continuity of ξ_i at $a \in \Omega_+$ with $a_i > 0$. Suppose $\xi_i(a) = x_i$ and $\varepsilon > 0$.

We can assume that ε is sufficiently small to be less than x_i and to provide, for the solution $y(x)$ to (13), the inequalities $y^{(i)}(x - \varepsilon) > 0$ on $[0, x_i - \varepsilon]$ and $y^{(i)}(x_i + \varepsilon) < 0$. In this case the point a has a neighborhood

$U \subset \Omega_+$ such that the above inequalities are satisfied for all solutions to (13) with initial data $a' \in U$. Hence, $|\xi_i(a') - x_i| < \varepsilon$. Continuity of ξ_i at $a \in \Omega_+$ with $a_i > 0$ is proved.

In the same way it is proved at $a \in \Omega_+$ with $a_i < 0$. Since $a_i \neq 0$ if $a \in K_j, i \neq j$, we have proved continuity of the restriction $\xi_i|_{K_j}$ in the case $i \neq j$.

As for $\xi_i|_{K_i}$, note that between two zeros of $y^{(i)}(x)$ there exists a zero x_j of another derivative $y^{(j)}(x)$. The values $y^{(m)}(x_j)$, $m = 0, 1, 2, 3$, due to continuity of $\xi_j|_{K_i}$, depend continuously on $a \in K_i$, whereas the restriction $\xi_i|_{K_j}$ depends continuously on these values. This proves continuity of the restriction $\xi_i|_{K_i}$. ►

Lemma 8. *For any $k > 1$ there exist $Q > q > 1$ such that for any typical solution $y(x)$ to equation* (1) *the values of all expressions*

$$\left|\frac{y(x'''_{j+1})}{y(x'_j)}\right|^{\frac{1}{4}}, \qquad \left|\frac{y(x''_j)}{y(x'''_j)}\right|^{\frac{1}{4}}, \qquad \left|\frac{y(x'_j)}{y(x''_j)}\right|^{\frac{1}{4}},$$

$$\left|\frac{y'(x_j)}{y'(x''_j)}\right|^{\frac{1}{k+3}}, \qquad \left|\frac{y'(x'''_{j+1})}{y'(x_j)}\right|^{\frac{1}{k+3}}, \qquad \left|\frac{y'(x''_j)}{y'(x'''_j)}\right|^{\frac{1}{k+3}},$$

$$\left|\frac{y''(x'_j)}{y''(x'''_j)}\right|^{\frac{1}{2k+2}}, \qquad \left|\frac{y''(x_j)}{y''(x'_j)}\right|^{\frac{1}{2k+2}}, \qquad \left|\frac{y''(x'''_{j+1})}{y''(x_j)}\right|^{\frac{1}{2k+2}},$$

$$\left|\frac{y'''(x''_{j+1})}{y'''(x_j)}\right|^{\frac{1}{3k+1}}, \qquad \left|\frac{y'''(x'_j)}{y'''(x_j)}\right|^{\frac{1}{3k+1}}, \qquad \left|\frac{y'''(x_j)}{y'''(x'_j)}\right|^{\frac{1}{3k+1}}$$

with sufficiently large j are contained in the segment $[q, Q]$.

◄ Let us define the continuous functions $\psi_{ijl} : K_i \to \mathbb{R}$ (all indices i, j, l are from 0 to 3 and pairwise different) taking each point $a \in K_i$ to the ratio of the absolute values of the j-th derivative of the solution $y(x)$ to (13) at 0 and at the next point where the l-th derivative vanishes, i.e. $\psi_{ijl}(a) = \left|\dfrac{a_j}{y^{(j)}(\xi_l(a))}\right|$ (both the numerator and the denominator are non-zero if $a \in K_i$).

Due to Lemma 6, each function ψ_{ijl} at all points of the compact set K_i is positive and less than 1. Hence $0 < \inf\limits_{K_i} \psi_{ijl}(a) \leq \sup\limits_{K_i} \psi_{ijl}(a) < 1$.

Now consider an arbitrary typical solution $y(x)$ to (1) and two its nodes, say x'_j and x'''_{j+1}, with sufficiently large numbers such that the related points in S^3 belong to Ω^1_+. In this case we can choose constants $A \neq 0$ and $B > 0$ such that the function $z(x) = Ay(Bx + x'_j)$ is a solution to (13) with $a \in K_1$. Indeed, this is equivalent to existence of $A \neq 0$ and $B > 0$ such that

$$|A|^{k-1} = B^4 p_0;$$

$$\sum_{m=0,2,3} \left(AB^m y^{(m)}(x_j')\right)^2 = 1,$$

which follows from existence of a root A to the equation

$$\left(y(x_j')\right)^2 A^2 + \left(y''(x_j')\right)^2 p_0^{-1} |A|^{k+1} + \left(y'''(x_j')\right)^2 p_0^{-\frac{3}{2}} |A|^{\frac{3k+1}{2}} = 1.$$

The value $\left|\dfrac{y(x_{j+1}''')}{y(x_j')}\right|^{\frac{1}{4}}$ is equal to this for $z(x)$ at $\xi_3(a)$ and 0, where $a_0 = |A|$, $a_1 = 0$, $a_2 = |A| B^2$, $a_3 = |A| B^3$, i.e. equal to $\psi_{103}(a)^{-\frac{1}{4}}$. Put $q = \left(\sup\limits_{K_1} \psi_{103}(a)\right)^{-\frac{1}{4}}$, $Q = \left(\inf\limits_{K_1} \psi_{103}(a)\right)^{-\frac{1}{4}}$ and obtain the statement of the lemma for the first ratio. The same procedure can be used for others. Then we just choose the minimum of 12 values of q and the maximum of 12 values of Q. ►

Lemma 9. *The domain of any typical* (*to the right*) *solution* $y(x)$ *to equation* (1) *is right-bounded. If* x^* *is its right boundary, then*

$$\overline{\lim_{x \to x^*}} \left|y^{(n)}(x)\right| = +\infty, \qquad n = 0, 1, 2, 3. \tag{14}$$

◄ It follows from Lemma 8 that the absolute values of the neighboring local extrema of any typical solution for sufficiently large number, say for $j \geq J$, satisfy the inequality $\left|y(x_{j+1}')\right| \geq q^{12} \left|y(x_j')\right|$ with some $q > 1$, whence

$$\left|y(x_j')\right| \geq q^{12(j-J)} \left|y(x'_J)\right|. \tag{15}$$

In particular, this yields (14) for $n = 0$. Other n are treated similarly.

It is proved in [7] that there exists a constant $C > 0$ depending only on k and p_0 such that all positive solutions to equation (1) defined on a segment $[a, b]$ satisfy the inequality $|y(x)| \leq C |b - a|^{-\frac{4}{k-1}}$. The same holds for negative ones. Hence the local extrema satisfy the estimate $\left|y(x_j')\right| \leq C \left(x_j - x_{j-1}\right)^{-\frac{4}{k-1}}$, which yields, together with (15), the inequality $(x_j - x_{j-1}) \leq Q^{-3(k-1)(j-J)} \left|\frac{C}{y(x'_J)}\right|^{\frac{k-1}{4}}$.

It follows from $Q > 1$ that $Q^{-3(k-1)} < 1$, $\sum\limits_{j=J}^{\infty} (x_j - x_{j-1}) < \infty$, and the domain is right-bounded. ►

Lemma 10. *For any* $k > 1$ *there exists positive constants* $m \leq M$ *such that for any typical solution* $y(x)$ *to equation* (1) *the distance between* *its neighboring points of local extremum, x'_j and x'_{j+1}, ultimately satisfies the estimates*

$$m \le (x'_{j+1} - x'_j) F_y(x'_j)^{k-1} \le M. \tag{16}$$

◀ Put $E_+ = \Phi_E\left(\Omega^1_+\right)$. It is a compact subset of the set E defined by (6) and lying inside Ω_+. Put

$$m = \inf\left\{\xi_1(a) : \ a \in E_+, a_1 = 0\right\} > 0;$$

$$M = \sup\left\{\xi_1(a) : \ a \in E_+, a_1 = 0\right\} < \infty.$$

Let $y(x)$ be a typical solution to equation (1), x'_j and x'_{j+1} be neighboring points of its local extremum. We can choose positive constants A and B such that the function $z(x) = Ay(Bx+x'_j)$ is a solution to equation (1) with $p_0 = 1$ and its data at zero correspond to some point in E, i.e. $F_z(0) = 1$. It is sufficient for this to find a positive solution to the system

$$A^{k-1} = B^4 p_0;$$

$$\sum_{m=0}^{3} \left|AB^m y^{(m)}(x'_j)\right|^{\frac{1}{m(k-1)+4}} = 1,$$

namely

$$A = \left(\sum_{m=0}^{3} \left|\frac{y^{(m)}(x'_j)}{p_0^{\frac{m}{4}}}\right|^{\frac{1}{m(k-1)+4}}\right)^{-4};$$

$$B = \left(\sum_{m=0}^{3} \left|\rho y^{(m)}(x'_j)\right|^{\frac{1}{m(k-1)+4}}\right)^{-(k-1)} = F_y(x_j)^{-(k-1)}.$$

Moreover, for local extrema with sufficiently large numbers, the point defined in $\mathbb{R}^4$ by the data of the function $z(x)$ at zero belongs to E_+. Hence the first positive point L of local extremum of $z(x)$ belongs to $[m, M]$, whence the difference $x'_{j+1} - x'_j$ is equal to LB and satisfies (16). ▶

Lemma 11. *For any $k > 1$ and $p_0 > 0$ there exists a constant $\theta > 0$ such that local extrema of any typical solution $y(x)$ to equation (1), ultimately satisfy the inequality $\left|y(x'_j)\right| \ge \theta F_y(x'_j)^4$.*

◀ Let $y(x)$ be a typical solution to equation (1) and x'_j be its local extremum point with sufficiently large number. Put $\theta = \inf\{|a_0| : a \in E_+, a_1 = 0\} > 0$ and choose a constant $\lambda > 0$ such that the data at zero for the solution $z(x) = \lambda^4 y(\lambda^{k-1}x + x'_j)$ correspond to some point in E_+. Then $F_z(0) = 1$ and $|z(0)| \ge \theta$. Since $z(0) = \lambda^4 y(x'_j)$ and $F_z(0) = \lambda F_y(x'_j)$, the lemma is proved. ▶

Remark 1. *For typical solutions to* (1) *with their corresponding curves lying completely in Ω_+, the statements of Lemmas 8, 10, and 11 hold in the whole domain, not only ultimately.*

Theorem 1. *For any real $k > 1$ and $p_0 > 0$ there exist positive constants C_1 and C_2 such that local extrema of any typical maximally extended to the right solution $y(x)$ to equation (1) in some neighborhood of the right bound x^* of its domain satisfy the inequalities* $C_1(x^* - x'_j)^{-\frac{4}{k-1}} \le \le \left|y(x'_j)\right| \le C_2(x^* - x'_j)^{-\frac{4}{k-1}}$.

◀ Let x'_J and x'_{J+1} be two neighboring points of local extremum of a solution $y(x)$ such that the statements of Lemmas 8, 10, and 11 hold. According to these Lemmas, for all $j \ge J$ we have

$$x'_{j+1} - x'_j \le MF_y(x'_j)^{-(k-1)} \le MF_y(x'_J)^{-(k-1)} q^{-3(k-1)(j-J)},$$

which implies $x^* - x'_J = \sum_{j=J}^{\infty} \left(x'_{j+1} - x'_j\right) \le \dfrac{MF_y(x'_J)^{-(k-1)}}{1 - q^{-3(k-1)}}$ and

$$\left|y(x'_J)\right| (x^* - x'_J)^{\frac{4}{k-1}} \le \frac{F_y(x'_J)^4}{\rho} \left(\frac{MF_y(x'_J)^{-(k-1)}}{1 - q^{-3(k-1)}}\right)^{\frac{4}{k-1}} = \\ = \left(\frac{Mp_0^{-\frac{1}{4}}}{1 - q^{-3(k-1)}}\right)^{\frac{4}{k-1}}.$$

On the other hand, $x'_{j+1} - x'_j \ge mF_y(x'_J)^{-(k-1)} Q^{-3(k-1)(j-J)}$, which implies

$$\left|y(x'_J)\right| (x^* - x'_J)^{\frac{4}{k-1}} \ge \theta F_y(x'_J)^4 \left(\frac{mF_y(x'_J)^{-(k-1)}}{1 - Q^{-3(k-1)}}\right)^{\frac{4}{k-1}} = \\ = \theta \left(\frac{m}{1 - Q^{-3(k-1)}}\right)^{\frac{4}{k-1}}. \ ▶$$

Asymptotic Classification of the Solutions to the Fourth-Order Equation (1). In this part we consider the asymptotic behavior of nontrivial solutions to equation (1) in the cases not previously considered. Then asymptotic classification of all maximally extended solutions to equation (1) will be given.

First for solutions to equation (1) generating in $\mathbb{R}^4$ curves lying entirely in Ω_+, we describe their asymptotic behavior near the left boundary of the domain.

Lemma 12. *Suppose $y(x)$ is a maximally extended to the left nontrivial solution to equation* (1) *with derivatives changing their signs according to scheme* (7). *Then the domain of $y(x)$ is unbounded to the left, the functions $y(x), y'(x), y''(x), y'''(x)$ tend to zero as $x \to -\infty$, and the distance between its neighboring zeros tends monotonically to ∞ as $x \to -\infty$.*

Using the substitution $x \to -x$ we can describe the asymptotic behavior of non-typical solutions near the right boundaries of their domains. Combining these results we obtain the following theorem.

Theorem 2. *Suppose $k > 1$ and $p_0 > 0$. Then all maximally extended solutions to equation* (1) *are divided into the following four types according to their asymptotic behavior* (Fig. 3).

0. The trivial solution $y(x) \equiv 0$.

1. Oscillatory solutions defined on $(-\infty, b)$. The distance between their neighboring zeros infinitely increases near the left boundary of the domain and tends to zero near the right one. The solutions and their derivatives satisfy the relations $\lim\limits_{x\to-\infty} y^{(j)}(x) = 0$, $\overline{\lim\limits_{x\to b}} \left|y^{(j)}(x)\right| = \infty$ for $j = 0, 1, 2, 3$. At the points of local extremum the following estimates hold:

$$C_1 \left|x - b\right|^{-\frac{4}{k-1}} \le \left|y(x)\right| \le C_2 \left|x - b\right|^{-\frac{4}{k-1}} \tag{17}$$

with the positive constants C_1 and C_2 depending only on k and p_0.

2. Oscillatory solutions defined on $(b, +\infty)$. The distance between their neighboring zeros tends to zero near the left boundary of the domain and infinitely increases near the right one. The solutions and their derivatives satisfy the relations $\lim\limits_{x\to+\infty} y^{(j)}(x) = 0$, $\overline{\lim\limits_{x\to b}} \left|y^{(j)}(x)\right| = \infty$ for $j = 0, 1, 2, 3$. At the points of local extremum estimates (17) hold with the positive constants C_1 and C_2 depending only on k and p_0.

3. Oscillatory solutions, defined on bounded intervals (b', b''). All their derivatives $y^{(j)}$, with $j = 0, 1, 2, 3, 4$ satisfy $\overline{\lim\limits_{x\to b'}} \left|y^{(j)}(x)\right| = \overline{\lim\limits_{x\to b''}} \left|y^{(j)}(x)\right| = \infty$. At the points of local extremum sufficiently close to any boundary of the domain, estimates (17) hold respectively with $b = b'$ or $b = b''$ and the positive constants C_1 and C_2 depending only on k and p_0.

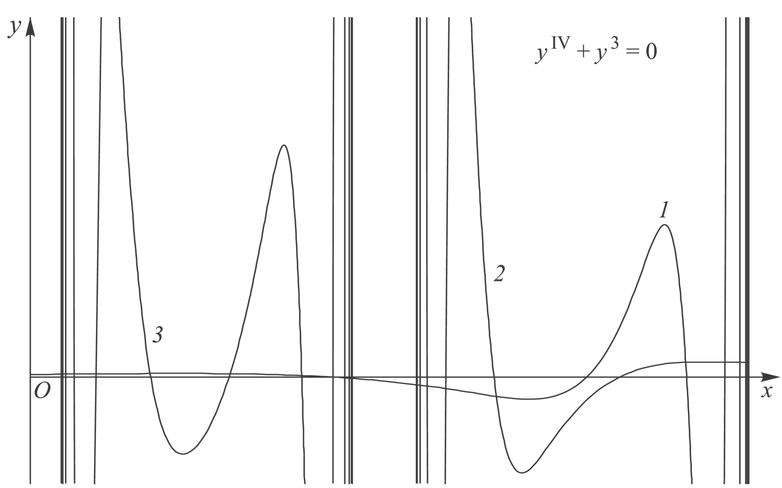

Fig. 3. Solutions to equation (1)

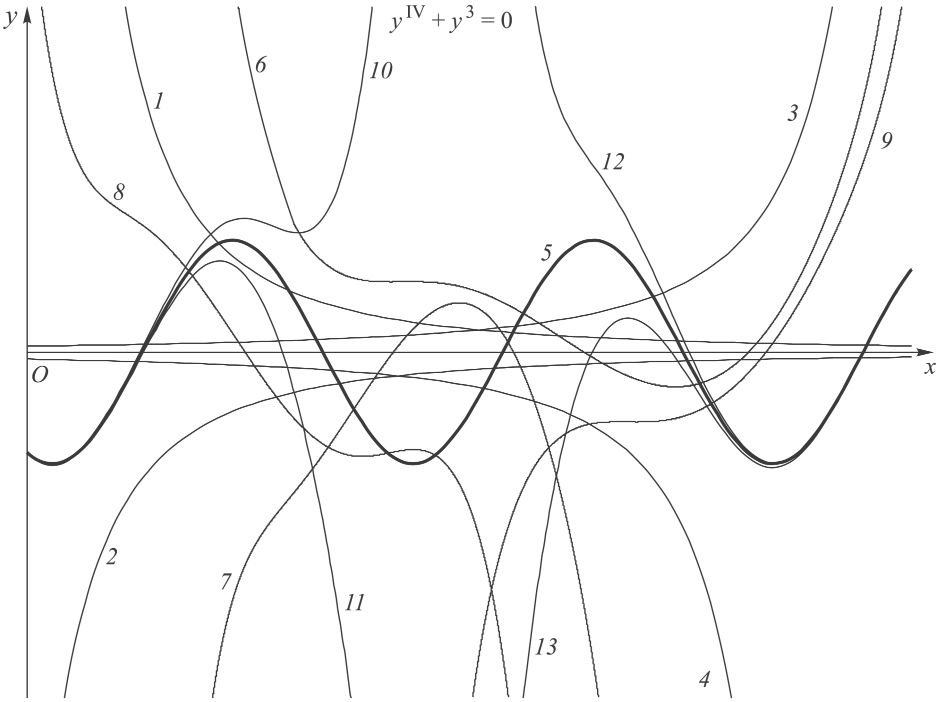

Fig. 4. Solution to equation (2)

Asymptotic Classification of the Solutions to the Fourth-Order Equation (2). In this section previously obtained results on the asymptotic behavior of solutions to equation (2) are formulated [7, 28].

Theorem 3. *Suppose $k > 1$ and $p_0 > 0$. Then all maximally extended solutions to equation* (2) *are divided into the following fourteen types according to their asymptotic behavior* (Fig. 4).

0. The trivial solution $y(x) \equiv 0$.

1–2. Defined on $(b, +\infty)$ Kneser (up to the sign) solutions (see definition in [5]) with the power asymptotic behavior near the boundaries of the domain (with the relative signs $\pm$):

$$y(x) \sim \pm C_{4k}(x-b)^{-\frac{4}{k-1}}, \quad x \to b+0;$$

$$y(x) \sim \pm C_{4k}x^{-\frac{4}{k-1}}, \qquad x \to +\infty,$$

where $C_{4k} = \left(\dfrac{4(k+3)(2k+2)(3k+1)}{p_0(k-1)^4}\right)^{\frac{1}{k-1}}$.

3–4. Defined on semi-axes $(-\infty, b)$ Kneser (up to the sign) solutions with the power asymptotic behavior near the boundaries of the domain (with the relative signs $\pm$):

$$y(x) \sim \pm C_{4k}\,|x|^{-\frac{4}{k-1}}, \qquad x \to -\infty;$$

$$y(x) \sim \pm C_{4k}(b-x)^{-\frac{4}{k-1}}, \quad x \to b-0.$$

5. Defined on the whole axis periodic oscillatory solutions. All of them can be received from one, say $z(x)$, by the relation $y(x) = \lambda^4 z(\lambda^{k-1}x + + x_0)$ with arbitrary $\lambda > 0$ and x_0. So, there exists such a solution with any maximum $h > 0$ and with any period $T > 0$, but not with any pair (h, T).

6–9. Defined on bounded intervals (b', b'') solutions with the power asymptotic behavior near the boundaries of the domain (with the independent signs $\pm$):

$$y(x) \sim \pm C_{4k}(p(b'))(x - b')^{-\frac{4}{k-1}}, \qquad x \to b' + 0;$$

$$y(x) \sim \pm C_{4k}(p(b''))\,(b'' - x)^{-\frac{4}{k-1}}, \quad x \to b'' - 0.$$

10–11. Defined on semi-axes $(-\infty, b)$ solutions which are oscillatory as $x \to -\infty$ and have the power asymptotic behavior near the right boundary of the domain: $y(x) \sim \pm C_{4k}(p(b))(b - x)^{-\frac{4}{k-1}}$, $x \to b - 0$. For each solution a finite limit of the absolute values of its local extrema exists as $x \to -\infty$.

12–13. Defined on semi-axes $(b, +\infty)$ solutions which are oscillatory as $x \to +\infty$ and have the power asymptotic behavior near the left boundary of the domain: $y(x) \sim \pm C_{4k}(p(b))(x - b)^{-\frac{4}{k-1}}$, $x \to b + 0$. For each solution a finite limit of the absolute values of its local extrema exists as $x \to +\infty$.

Asymptotic classification of the solutions to the third-order equation (3). In this section previously obtained results on the asymptotic behavior of solutions to equation (3) are formulated [7, 28].

Theorem 4. *Suppose $k > 1$, and $p(x)$ is a globally defined positive continuous function with positive limits p_* and p^* as $x \to \pm\infty$. Then any nontrivial non-extensible solution to* (3) *is either* (*Fig. 5*):

1–2) a Kneser solution on a semi-axis $(b, +\infty)$ satisfying

$$y(x) = \pm C_{3k}(p(b))\,(x - b)^{-\frac{3}{k-1}}(1 + o(1)) \quad \text{as} \quad x \to b + 0,$$

$$y(x) = \pm C_{3k}(p^*)\,x^{-\frac{3}{k-1}}(1 + o(1)) \qquad \text{as} \quad x \to +\infty,$$

where $C_{3k}(p) = \left(\dfrac{3(k+2)(2k+1)}{p(k-1)^3}\right)^{\frac{1}{k-1}}$;

3) an oscillating, in both directions, solution on a semi-axis $(-\infty, b)$ satisfying, at its local extremum points,

$$|y(x')| = |x'|^{-\frac{3}{k-1}+o(1)} \qquad \text{as} \quad x' \to -\infty,$$

$$|y(x')| = |b - x'|^{-\frac{3}{k-1}+o(1)} \quad \text{as} \quad x' \to b + 0;$$

4–5) an oscillating near the right boundary and non-vanishing near the left one solution on a bounded interval (b', b'') satisfying

$$y(x) = \pm C_{3k}(p(b'))(x-b')^{-\frac{3}{k-1}}(1+o(1))$$

as $x \to b'+0$, *and, at its local extremum points,*

$$|y(x')| = |b''-x'|^{-\frac{3}{k-1}+o(1)}$$

as $x' \to b''-0$.

Conclusion. Note that oscillatory solutions of equations (1) and (3) defined on $(-\infty; x_*)$ or $(x_*; +\infty)$, are the solutions of the form

$$y(x) = |p_0|^{-\frac{1}{k-1}}|x-x_*|^{-\alpha}h(\log|x-x_*|), \quad \alpha = \frac{n}{k-1}, \tag{18}$$

with $n=4$ and $n=3$ respectively and an oscillatory periodic function $h: \mathbb{R} \to \mathbb{R}$.

Indeed more general result takes place. Thus, for the equation

$$y^{(n)} + p_0\,|y|^{k-1}\,y(x) = 0, \quad n > 2, \quad k \in \mathbb{R}, \quad k > 1, \quad p_0 \neq 0, \tag{19}$$

the existence of oscillatory solutions of the type (18) is proved.

Theorem 5. *For any integer* $n > 2$ *and real* $k > 1$ *there exists a non-constant oscillatory periodic function* $h(s)$ *such that for any* $p_0 > 0$ *and* $x^* \in \mathbb{R}$ *the function*

$$y(x) = p_0^{-\frac{1}{k-1}}(x^*-x)^{-\alpha}h(\log(x^*-x)), \; -\infty < x < x^*, \; \alpha = \frac{n}{k-1}, \tag{20}$$

is a solution to equation (19).

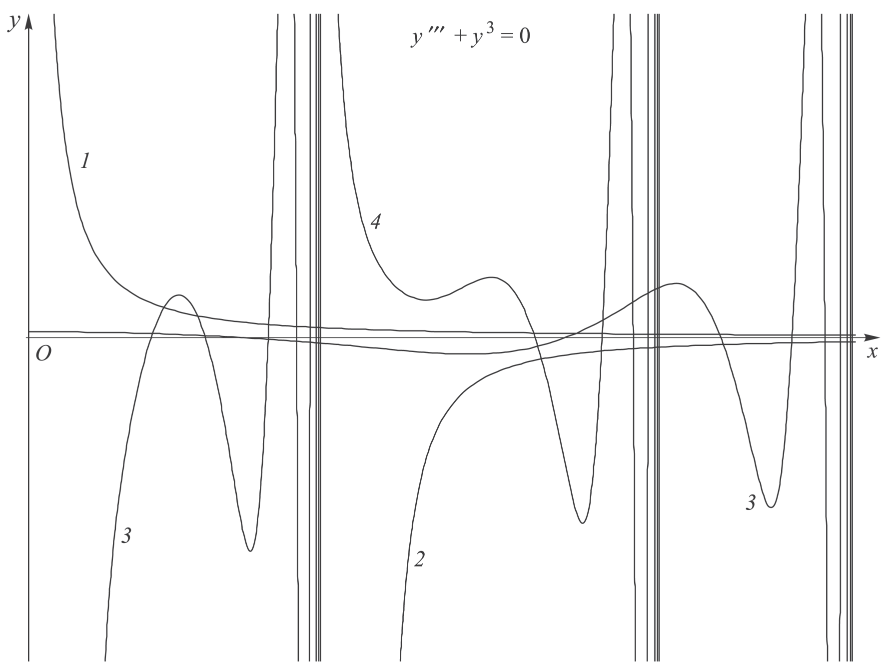

Fig. 5. Solution to equation (3)

Corollaries from this theorem for even and odd n are also proved for solutions defined near $+\infty$ [42].

This work was supported by the RFBR Grant (no. 11-01-00989).

REFERENCES

[1] Emden R. Gaskugeln. Leipzig, 1907.
[2] Zeldovich Ya.B., Blinnikov S.I., Shakura N.I. Physical foundations of the structure and evolution of stars. Moscow, MSU Publ., 1981.
[3] Bellman R. Stability Theory of Solutions of Differential Equations (Russ. translation), Moscow, 1954.
[4] Sansone J. Ordinary differential equations, vol. 2. Moscow, InLit Publ., 1954.
[5] Kiguradze I.T., Chanturia T.A. Asymptotic properties of solutions of nonautonomous ordinary differential equations. Dordrecht–Boston–London Kluwer, Academic Publishers, 1993.
[6] Astashova I.V., Filinovskii A.V., Kondratiev V.A., Muravei L.A. Some problems in the qualitative theory of differential equations. *J. of Natural Geometry*. Jnan Bhawan. London, 2003, vol. 23, no. 1–2, pp. 1–126.
[7] Astashova I.V. Qualitative properties of solutions to quasilinear ordinary differential equations. In: Astashova I.V. (ed.) Qualitative Properties of Solutions to Differential Equations and Related Topics of Spectral Analysis: scientific edition. Moscow, UNITY-DANA Publ., 2012, pp. 22–290.
[8] Atkinson F.V. On second order nonlinear oscillations. *Pac. J. Math.*, 1955, vol. 5, no. 1, pp. 643–647.
[9] Kiguradze I.T. Asymptotic properties of solutions of a nonlinear Emden – Fowler type differential equation. *Izv. Akad. Nauk SSSR, Ser. Mat*. [Math. USSR–Izvestija], 1965, vol. 29, no. 5, pp. 965–986 (in Russ.).
[10] *Waltman P.* Oscillation criteria for third order nonlinear differential equations. *Pac. J. Math.*, 1966, vol. 18, pp. 385–389.
[11] Kiguradze I.T. On monotone solutions of nonlinear ordinary nth-order differential equations. *Izv. Akad. Nauk SSSR, Ser. Mat*. [Math. USSR-Izvestija], 1969, vol. 6, pp. 1373–1398 (in Russ.).
[12] Kostin A.V. On asymptotic of non-extendable solutions to Emden – Fowler type equations. *DAN SSSR*, 1971, vol. 200, no. 1, pp. 28–31 (in Russ.).
[13] Kusano T., Naito M. Nonlinear oscillation of fourth-order differential equations. *Canad. J. Math*., 1976, no. 28(4), pp. 840–852.
[14] Lovelady D.L. An oscillation criterion for a fourth-order integrally superlinear differential equation. *Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Natur*., 1975, vol. (8) 58 (4), pp. 531–536.
[15] Kondratiev V.A., Samovol V.S. On certain asymptotic properties of solutions to equations of the Emden – Fowler type. *Differents. Uravn*. [Differential Equations], 1981, vol. 17, no. 4, pp. 749–750 (in Russ.).
[16] Kvinikadze G.G., Kiguradze I.T. On quickly growing solutions of nonlinear ordinary differential equations. *Soobsh. Academy of Science GSSR*, 1982, vol. 106, no. 3, pp. 465–468 (in Russ.).
[17] Taylor W.E. Jr. Oscillation criteria for certain nonlinear fourth order equations. *Internat. J. Math*. 1983, no. 6(3), pp. 551–557.
[18] Izobov N.A. On the Emden – Fowler equations with infinitely continuable solutions. *Mat. Zametki* [Mathematical Notes], 1984, vol. 35, iss. 2, pp. 189–199 (in Russ.).
[19] Kvinikadze G.G. On monotone regular and singular solutions of ordinary differential equations. *Differents. Uravn*. [Differential Equations], 1984, vol. 20, no. 2, pp. 360–361 (in Russ.).

[20] Astashova I.V. On asymptotic behavior of solutions of certain nonlinear differential equations. *UMN*, 1985. vol. 40, no. 5 (245), p. 197 (in Russ.).

[21] Astashova I.V. On asymptotic behavior of alternating solutions to certain nonlinear differential equations of the third and forth order. *Reports of extended session of a seminar of the I.N. Vekua Institute of Applied Mathematics*, Tbilisi, 1988, no. 3(3), pp. 9–12 (in Russ.).

[22] Kiguradze I.T. On the oscillation criteria for one class of ordinary differential equations *Differents. Uravn*. [Differential Equations], 1992, vol. 28, no. 2, pp. 207–219 (in Russ.).

[23] Chanturia T.A. On existence of singular and unbounded oscillatory solutions to Emden – Fowler Type Differential equations. *Differents. Uravn*. [Differential Equations], 1992, vol. 28, no. 6, pp. 1009–1022.

[24] Astashova I.V. On qualitatuve properties of solutions to Emden – Fowler type equations. *UMN*, 1996, vol. 51, no. 5, pp. 185 (in Russ.).

[25] Kozlov V.A. On Kneser solutions of higher order nonlinear ordinary differential equations. *Ark. Mat*., 1999, vol. 37, no. 2, pp. 305–322.

[26] Kiguradze I.T. On blow-up kneser solutions of nonlinear ordinary higher-order differential equations. *Differents. Uravn*. [Differential Equations], 2001, vol. 37, no. 6, pp. 735–743 (in Russ.).

[27] Kon'kov A.A. On solutions of nonautonomous ordinary differential equations. *Izv. Ross. Akad. Nauk, Ser. Mat*. [Izvestiya: Mathematics], 2001, vol. 65, no. 2, pp. 81–126 (in Russ.).

[28] Astashova I.V. Application of Dynamical Systems to the Study of Asymptotic Properties of Solutions to Nonlinear Higher-Order Differential Equations. *J. of Mathematical Sciences*. Springer Science + Business Media. 2005, no. 126(5), pp. 1361–1391.

[29] Astashova I.V. Classification of solutions of fourth-order equations of the Emden – Fowler type. *Differents. Uravn*. [Differential Equations], 2008, vol. 44, no. 6, pp. 881–882 (in Russ.).

[30] Astashova I.V. Uniform estimates for positive solutions of higher-order quasilinear differential equations. *Proceedings of Steklov Mathematical Institute*, 2008, vol. 261, iss. 1, pp. 22–33.

[31] Astashova I.V. Asymptotic classification of solutions to 3rd and 4th Order Emden – Fowler type differential equations. *Proceeding of the International Conference*. Euler International Mathematical Institute, 2011. EIMI, St. Petersburg, 2011, pp. 9–12.

[32] Astashova I.V. Uniform estimates for solutions to the third-order Emden – Fowler Type autonomous differential equation. *Functional differential equations*, vol. 18, 1–2. Ariel University Center of Samaria. Ariel, Israel, 2011, pp. 5–63.

[33] Bartušek M., Došlá S. Asymptotic problems for fourth-order nonlinear differential equations. *Boundary Value Problems*, 2013, 2013:89. DOI:10.1186/1687-2770-2013-89

[34] Astashova I.V. On power and non-power asymptotic behavior of positive solutions to Emden – Fowler type higher-order equations. *Advances in Difference Equations*, 2013. DOI: 10.1186/10.1186/1687-1847-2013-220

[35] Astashova I.V. On asymptotic behavior of solutions to a forth-order nonlinear differential equation mathematical methods in finance and business administration. *Proceedings of the 1st WSEAS International Conference on Pure Mathematics* (PUMA '14), Tenerife, Spain, January 10–12, 2014, WSEAS Press, 2013, pp. 32–41.

[36] Bidaut-Véron M.F. Local and global behaviour of solutions of quasilinear elliptic equations of Emden – Fowler type. *Arch. Rat. Mech. Anal*., 1989, vol. 107, pp. 293–324.

[37] Kondratiev V.A. On qualitative properties of solutions to semi-linear elliptic equations. *Trudy of I.G. Petrovskiy seminar*, 1991, vol. 16, pp. 186–190 (in Russ.).

[38] Egorov Yu.V., Kondratiev V.A., Oleinik O.A. Asymptotic behavior of the solutions to nonlinear elliptic and parabolic systems in tube domains. *Mat. Sbornik* [Sbornik: Mathematics], 1998, 189:3, pp. 45–68 (in Russ.).

[39] Mitidieri E., Pohozhaev S.I. A priori estimates and the absence of positive solutions to non-linear partial differential equalities and inequalities. *Proceedings of Steklov Mathematical Institute*, 2001, vol. 234, 383 p.

[40] Grishina G.V. On localization of support and unrealizable conditions of growth of solutions to semi-linear elliptic second-orderdifferential equations in unbounded domains. *Vestn. Mosk. Gos. Tekh. Univ. im. N.E. Baumana, Estestv. Nauki* [Herald of the Bauman Moscow State Tech. Univ., Nat. Sci.], 2012, no. 1, pp. 15–19 (in Russ.).

[41] Kondratiev V.A. On oscillation of solutions to linear third- and fourth-order equations. *Trudy MMO*, 1959, vol. 8, pp. 259–281 (in Russ.).

[42] Astashova I.V. On existence of quasi-periodic oscillatory solutions of Emden – Fowler type higher-order equations. *Differents. Uravn*. [Differential Equations], 2014, vol. 50, no. 6, pp. 847–848 (in Russ.).

The original manuscript was received by the editors in 23.06.2014

Асташова Ирина Викторовна — д-р физ.-мат. наук, профессор кафедры дифференциальных уравнений механико-математического факультета МГУ им. М.В. Ломоносова, профессор кафедры высшей математики Московского государственного университета экономики, статистики и информатики (МЭСИ). Автор 114 научных работ, в том числе трех монографий, в области качественной теории обыкновенных дифференциальных уравнений.

МГУ им. М.В. Ломоносова, Российская Федерация, 119991, Москва, Ленинские горы, д. 1.

МЭСИ, Российская Федерация, 119501, Москва, ул. Нежинская, д. 7.

Astashova I.V. — Dr. Sci. (Phys.-Math.), professor of Differential Equations department, Mechanical-Mathematical Faculty of the Lomonosov Moscow State University, professor of the Higher Mathematics department of the Moscow State University of Economics, Statistics and Informatics (MESI). Author of 114 publications, including 3 monographs, in the field of qualitative theory of ordinary differential equations.

Lomonosov Moscow State University, Leninskie Gory 1, Moscow, 119991 Russian Federation.

Moscow State University of Economics, Statistics and Informatics (MESI), Nezhinskaya ul. 7, Moscow, 119501 Russian Federation.