UDC 521 DOI: 10.18698/1812-3368-2019-2-4-16

THE PHYSICAL LIBRATIONS OF THE MOON CAUSED BY ITS TIDAL DEFORMATIONS

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Abstract

The Moon, like Earth, is not completely solid, and experiences deformation changes, for example due to the tides, caused by the gravitational pull of the Earth's orbit in a complex and resonant nature of the motion of the Moon. It is shown that these deformations lead to temporary variations of Moon inertia tensor components and consequently to the variations in the movement of the poles of the Moon, as well as to the variations of axial rotation. The indicated variations module is in the order of 10–12 mas (millisecond of arc). There variations are important for the development of the high-precision theory of lunar physical libration, suitable for modern projects for the reclamation of the Moon, in particular the Japanese project *ILOM*, which contemplates installing the telescope on the lunar surface and determining its orientation accuracy of the order of 1–0.1 msd, as well as the Russian lunar program, providing the launch of five automatic stations to the Moon in 2019–2024

Keywords

Physical librations of the Moon, tidal deformations, tidal effects, rotation of the Moon, selenopotential

Received 26.03.2018

The research was carried out with the financial support of the Russian Science Foundation (RSF grant no. 17-71-10254)

Introduction. The research of the physical librations of the Moon caused by its tidal deformations has been performed. The gravitational field of the Moon is considered in the framework of the classical Newton theory. The study is based on the Liouville equations and the equations of motion of a weakly deformable celestial body in Andoyer variables [1, 2]. In this case, the values of temporal (tidal) variations of the coefficients of the second harmonic of the selenopotential, given in Ref. [3], are used. The solution was found by the perturbation method and is presented in analytical form in Andoyer variables;

expressions for the variations of the angular velocity components were also obtained. The amplitudes and periods of physical librations of the Moon caused by its tidal deformations were estimated.

The solution of the Moon's librations task was obtained using two coordinate systems associated with this celestial body $C\xi_K\eta_K\zeta_K$ and $C\xi\eta\zeta$. The axes of these coordinate systems correspond to the principal central axes of inertia of the Moon, but to different values of the moments of inertia. The corresponding axes of these two coordinate systems coincide and there are simple relations between them: $C\xi_K = -C\xi$, $C\eta_K = C\zeta$, $C\zeta_K = C\eta$, which allow establishing relations between two sets of selenopotential constants, defined in the considered coordinate systems. Here, the parameters with the "*K*" index correspond to the classical selenographic coordinate system $C\xi_K\eta_K\zeta_K$, and the parameters without an index, to the principal axes of inertia used in solving the Liouville equations in Andoyer variables. For the principle moments of inertia, these relations have the form $A_K = A$, $B_K = C$ and $C_K = B$, and for the constants of the second harmonic the selenopotential

$$J_2^K = -\frac{1}{2}\left(J_2 - 6C_{22}\right);\ C_{22}^K = \frac{1}{4}\left(J_2 + 2C_{22}\right);$$
$$S_{22}^K = -\frac{1}{2}C_{21};\ \ C_{21}^K = 2S_{22};\ \ S_{21}^K = -S_{21}. \tag{1}$$

The inverse formulas of the specified transformation (1) take the form

$$J_2 = -\frac{1}{2}\left(J_2^K - 6C_2^K\right);\ \ C_{22} = \frac{1}{4}\left(J_2^K + 2C_{22}^K\right);$$
$$C_{21} = -2S_{22}^K;\ \ S_{22} = \frac{1}{2}C_{21}^K;\ \ S_{21} = -S_{21}^K. \tag{2}$$

The relations (1) and (2) are used in interpreting the obtained decision on the librations of the Moon due to its tidal deformations. We'll obtain the solution using the parameters (2) without the "*K*" index, and then convert the resulting solution to the parameters (1) for the classical (basic) selenographic coordinate system $C\xi_K\eta_K\zeta_K$.

Formulation of the problem. Due to the rotational deformation of the Moon, the polar moment of inertia of a celestial body increases by

$$\delta C_K = \frac{2}{9}k_2\frac{r_0^5\omega^2}{f},$$

and the equatorial moments of inertia decrease by the same magnitudes, but 2 times smaller in comparison with δC [3, 4]. Here ω is the angular velocity of rotation of the Moon; r_0 is the average radius of the Moon; k_2 is the Love number; f is the gravitational constant.

To describe the temporal variations of the selenopotential coefficients, let's introduce the dimensionless deformation parameter

$$D_\omega = \frac{\delta C_K}{m r_0^2} = \frac{2}{9} k_2 \frac{r_0^3 \omega^2}{mf},$$

where m is mass of the Moon. At the same time, the elastic Moon rotates as an absolutely rigid body [1, 2], but with modified inertia moments

$$A = A_K + 3D_\omega m r_0^2;\; B = B_K + 3D_\omega m r_0^2;\; C = C_K. \tag{3}$$

Here, A_K, B_K, C_K are the principle moments of inertia of the Moon, which are determined by observations and can be expressed in terms of the mean values of the selenopotential coefficients. If the dimensionless polar moment of inertia I would be introduced by the formula $C_K = I m r_0^2$, then the expressions for the moments of inertia can be obtained from formulas (1)–(3):

$$A_K = \left(I + C_{20}^K - 2C_{22}^K\right) m r_0^2;\; B_K = \left(I + C_{20}^K + 2C_{22}^K\right) m r_0^2;\; C = I m r_0^2. \tag{4}$$

The expressions for variations of moments of inertia of the Moon can be written if the classical condition for temporal variations of axial moments of inertia $\delta A_K + \delta B_K + \delta C_K = 0$ is fulfilled (it corresponds to subsurface mass redistribution) for a changeable celestial body

$$\begin{gathered}\frac{\delta A}{C} = -\frac{2}{I}\delta C_{22} - \frac{1}{3I}\delta J_2;\quad \frac{\delta B}{C} = \frac{2}{I}\delta C_{22} - \frac{1}{3I}\delta J_2;\quad \frac{\delta C}{C} = \frac{2}{3I}\delta J_2;\\ \frac{\delta F}{C} = \frac{2}{I}\delta S_{22};\quad \frac{\delta E}{C} = \frac{1}{I}\delta C_{21};\quad \frac{\delta D}{C} = \frac{1}{I}\delta S_{21}.\end{gathered} \tag{5}$$

Tidal variations of the selenopotential coefficients were obtained on the basis of the solution of the classical tidal deformation problem of a celestial body [3]. These periodic variations, caused by the gravitational influence of the Earth, were presented in a standard form

$$\begin{gathered}\delta J_2^K = J_2^{(0.0.0.0;K)} + \sum_{|\mathbf{v}|\ge 1} J_2^{(\mathbf{v};K)} \cos\boldsymbol{\theta}_\mathbf{v};\quad \delta C_{22}^K = C_{22}^{(0.0.0.0;K)} + \sum_{|\mathbf{v}|\ge 1} C_{22}^{(\mathbf{v};K)} \cos\boldsymbol{\theta}_\mathbf{v};\\ \delta S_{22}^K = S_{22}^{(0.0.0.0;K)} + \sum_{|\mathbf{v}|\ge 1} S_{22}^{(\mathbf{v};K)} \sin\boldsymbol{\theta}_\mathbf{v};\quad \delta C_{21}^K = C_{21}^{(0.0.0.0;K)} + \sum_{|\mathbf{v}|\ge 1} C_{21}^{(\mathbf{v};K)} \sin\boldsymbol{\theta}_\mathbf{v};\\ \delta S_{21}^K = S_{21}^{(0.0.0.0;K)} + \sum_{|\mathbf{v}|\ge 1} S_{21}^{(\mathbf{v};K)} \cos\boldsymbol{\theta}_\mathbf{v},\end{gathered} \tag{6}$$

where $J_2^{(\mathbf{v};K)}$, $C_{22}^{(\mathbf{v};K)}$, …, $S_{21}^{(\mathbf{v};K)}$ are dimensionless amplitudes of the variation of the second harmonic coefficients of the selenopotential; $\boldsymbol{\theta}_\mathbf{v} = v_1 l_M + v_2 l_S + + v_3 F + v_4 D$ is linear combination of the four well-known arguments of the theory of the orbital motion of the Moon with integer coefficients. These arguments are linear functions of time and correspond to constant frequencies n_M, n_S, n_F and n_D [5, 6]. In series (6), the constant components of the tidal variations of the selenopotential coefficients are separated, due to the resonant nature of the movement $J_2^{(0.0.0.0;K)}$, $C_{22}^{(0.0.0.0;K)}$, …, $S_{21}^{(0.0.0.0;K)}$.

The numerical values of the amplitudes of variations (6) $J_2^{(\mathbf{v};K)}$, $C_{22}^{(\mathbf{v};K)}$, …, $S_{21}^{(\mathbf{v};K)}$ (up to 10^{-11} rad) are determined in Ref. [3] (the number of trigonometric terms are shown in parentheses): for coefficients J_2 (35); C_{22} (31); S_{22} (32); C_{21} (26); S_{21} (25). The maximum values of these variations are: for variations J_2 ($1.545\cdot10^{-8}$ with a period of 27.555 days); C_{22} ($0.722\cdot10^{-8}$ with a period of 27.555 days); S_{22} ($1.039\cdot10^{-8}$ with a period of 27.555 days); C_{21} ($2.221\cdot10^{-8}$ with a period of 27.212 days); S_{21} ($0.123\cdot10^{-8}$ with a period of 2190.4 days). In the present work, variations (6) are considered as given functions of time. The *goal* is to identify dynamic effects in the rotational motion of the Moon caused by these tidal variations (6). The equations of motion in Andoyer canonical variables (and their modifications) are used in the form, obtained in [2], for this purpose.

The equations of motion of the Liouville problem in Andoyer variables. We'll consider the Moon as a weakly deformable body, experiencing deformation due to its own rotation and under the influence of the gravitational attraction of the Earth. In other words, we will consider the Moon as a free body, but with a changeable form and with a time-varying internal structure.

Let's assume that the particles of the body during its movement either slightly deviate from their original positions, or shift in a predetermined manner in time with a small velocity. It can also be assumed that the body has an internal solid shell, with which some Cartesian coordinate system $C\xi\eta\zeta$ (body axis) is associated, and an external deformable shell. Let $Cxyz$ be the Cartesian coordinate system with the origin at the center of mass of the Moon, maintaining a constant orientation in space. The Andoyer variables will be introduced, which are related to the vector of the angular momentum of the rotational motion of the Moon $\mathbf{G}$ [1, 2]

$$G,\ \theta,\ \rho,\ l,\ g,\ h. \tag{7}$$

Here for brevity we will not give a detailed description of these variables, but only note that these variables determine the magnitude and orientation of the vector of the angular momentum of the body (Moon) $\mathbf{G}$ in two coordinate systems: 1) in the basic ecliptic coordinate system $Cxyz$; 2) in the intermediate coordinate system $CG_1G_2G_3$, associated with the vector $\mathbf{G}$ and in the axes associated with the heavenly body $C\xi\eta\zeta$.

In particular, the Andoyer variables θ, l, g are Euler angles that determine the orientation of the axes of inertia of the body in an intermediate coordinate system $CG_1G_2G_3$, axes of which retain their orientation in space (for an isolated celestial body).

Let's introduce the Andoyer variables L, G, H, where L is projection of the $\mathbf{G}$ vector on the polar axis of the body $C\zeta$; H is projection of the $\mathbf{G}$ vector on the axis Cz; G is the magnitude of the vector $\mathbf{G}$. It is obvious, that

$$L = G\cos\theta;\ G = |\mathbf{G}|;\ H = G\cos\rho. \tag{8}$$

The variables (8) and l, g, h are canonical, the equations of rotational motion of the considered weakly deformable body in these variables have a canonical form [1, 2]:

$$\begin{aligned} &\frac{dl}{dt} = \frac{\partial K}{\partial L},\ \frac{dL}{dt} = -\frac{\partial K}{\partial l};\\ &\frac{dg}{dt} = \frac{\partial K}{\partial G},\ \frac{dG}{dt} = -\frac{\partial K}{\partial g};\\ &\frac{dh}{dt} = \frac{\partial K}{\partial H},\ \frac{dH}{dt} = -\frac{\partial K}{\partial h}; \end{aligned} \tag{9}$$

$$\begin{aligned} K = \frac{1}{2}G^2\Big[\big(&a\sin^2 l + b\cos^2 l - d\sin 2l\big)\sin^2\theta + \\ &+ c\cos^2\theta - \sin 2\theta\big(e\sin l + f\cos l\big)\Big]. \end{aligned} \tag{10}$$

Here, by the condition of the problem, the coefficients a, b, …, f are known functions of time and are expressed in terms of axial and centrifugal moments of inertia by simple formulas

$$\begin{aligned} &a = \frac{BC - D^2}{\Delta};\ b = \frac{AC - E^2}{\Delta};\ c = \frac{AB - F^2}{\Delta};\\ &f = -\frac{ED + FC}{\Delta};\ e = -\frac{DF + BE}{\Delta};\ d = -\frac{FE + AD}{\Delta};\\ &\Delta = ABC - AD^2 - BE^2 - CF^2 - 2DEF. \end{aligned} \tag{11}$$

The Hamiltonian of (10), (11) can be represented as the sum of the zero-order (K_0) and perturbed (K_0) Hamiltonians. Zero-order Hamiltonian is defined as

$$K_0 = \frac{1}{2} G^2 \left[\left(\frac{1}{A_0} + \frac{1}{B_0} \right) \sin^2 \theta + \frac{1}{C_0} \cos^2 \theta \right], \tag{12}$$

or

$$K_0 = \frac{1}{2} G^2 \left[\left(\frac{1}{A_0} + \frac{1}{B_0} \right) \left(G^2 - L^2 \right) + \frac{1}{C_0} L^2 \right], \tag{13}$$

where A_0, B_0, C_0 are unperturbed values of axial moments of inertia (for example, corresponding to the undeformed state of the Moon).

The Hamiltonian K_0 corresponds to the undisturbed rotation of a solid axisymmetric body (or close in its dynamic structure to axisymmetric). In this case, the undisturbed rotational motion of the considered body is described by simple formulas

$$L = L_0 \ (\theta = \theta_0); \ l = n_l t + l_0; \ g = n_g t + g_0, \tag{14}$$

where z_0 is the initial value of the corresponding variable $z = (L, \theta, l, g)$, and the unperturbed frequencies of the Euler body motion are constant and are determined by the formulas

$$n_l = L_0 \left(\frac{1}{A_0} - \frac{1}{C_0} \right); \ n_g = \frac{1}{2} G_0 \left[\frac{1}{C_0} + \frac{1}{2} \left(\frac{1}{A_0} + \frac{1}{B_0} \right) \right]. \tag{15}$$

In the case of the Moon, we'll take its axial rotation around the polar axis of inertia (the axis of the greatest moment of inertia $C\eta$) as the undisturbed rotational motion. For this unperturbed motion $L_0 = 0$ and, accordingly, the frequency $n_l = 0$.

First-order perturbations in the rotation of the Moon due to its tidal deformations in Andoyer variables. The influence of perturbing factors on the rotation of a deformable body is characterized by a small parameter μ, which is proportional to the magnitude of the variation of the second harmonic coefficients of the selenopotential, i.e., for the Moon $\mu \simeq 10^{-8}$. Therefore, it is advisable to use method of the small parameter to solve the considered problem.

Keeping the terms of the first order of smallness, for the perturbing Hamiltonian (the terms of the first order of smallness μ) on the basis of formulas (10), (11), the expression will be obtained

$$K_1 = \frac{1}{2}\frac{G^2}{JC_0}\left[\sin^2\theta\left(\frac{1}{3}\left(6\delta C_{22}+\delta J_2\right)\sin^2 l+\frac{1}{3}\left(-6\delta C_{22}+\delta J_2\right)\cos^2 l+2\delta S_{22}\sin 2l\right)-\right.$$

$$\left.-\frac{2}{3}\delta J_2\cos^2\theta+\sin 2\theta\left(\delta C_{21}\sin l+\delta S_{21}\cos l\right)\right]. \qquad (16)$$

The selected perturbing Hamiltonian allows to study the dynamic effects in the rotation of a celestial body, caused by cyclical variations or secular changes of the main second harmonic coefficients of the selenopotential δJ_2, δC_{22}, δC_{21}, δS_{21} and δS_{22}. Thus, first-order disturbances of variables l, g and L (7), (8) are determined by simple quadratures

$$\delta l = \int \frac{\partial K_1}{\partial L}dt;\ \delta L = -\int \frac{\partial K_1}{\partial l}dt;\ \delta g = \int \frac{\partial K_1}{\partial G}dt. \qquad (17)$$

The partial derivatives in integrands (17) are calculated for the unperturbed values of the variables (14), (15) and are known functions of time. For brevity, let's omit the expressions for the indicated partial derivatives and present the results of integration by formulas (17)

$$\delta l = \frac{1}{J}\sum_{\|\mathbf{v}\|>0}\frac{\omega}{\omega_{\mathbf{v}}}S_{\mathbf{v}}^{(21)}\sin\boldsymbol{\theta}_{\mathbf{v}};\ \frac{\delta L}{G} = \frac{2}{J}\sum_{\|\mathbf{v}\|>0}\frac{\omega}{\omega_{\mathbf{v}}}S_{\mathbf{v}}^{(22)}\cos\boldsymbol{\theta}_{\mathbf{v}};\ \delta\theta = -\frac{\delta L}{G}; \qquad (18)$$

$$\delta g = \frac{\omega}{J}\sum_{\|\mathbf{v}\|>0}\frac{\omega}{\omega_{\mathbf{v}}}\left(\frac{1}{3}J_{\mathbf{v}}^{(2)}-2C_{\mathbf{v}}^{(22)}\right)\sin\boldsymbol{\theta}_{\mathbf{v}}. \qquad (19)$$

Considering the differential ratio $\delta L = -\sin\theta\delta\theta$, we obtain another expression for the variation of the variable θ:

$$\delta\theta = -\frac{2}{J}\sum_{\|\mathbf{v}\|>0}\frac{\omega}{\omega_{\mathbf{v}}}S_{\mathbf{v}}^{(22)}\cos\boldsymbol{\theta}_{\mathbf{v}}. \qquad (20)$$

This is a complete solution for first-order perturbations in the indicated variables. There will be additional terms in the projections of the angular velocity due to variations in the geometry of the masses of the Moon. There is no data on the variations of its relative kinetic moment (or its components along the axes of inertia of the Moon *P*, *Q* and *R*) for the Moon yet, and the corresponding perturbations in the rotation of the Moon are not considered in the article.

The variations of auxiliary coefficients of the geopotential were used in formulas (18)–(20). They are related to the axes of coordinates $O\xi\eta\zeta$, associated with the Moon and directed along the geocentric radius vector of the center of mass of the Moon (axis $O\xi$), along the polar axis of inertia $O\eta$ and tangential to the orbit, and in the opposite direction with respect to the direction of motion $O\zeta$. However, in selenodesy, the basic selenographic coordinate system $C\xi_K\eta_K\zeta_K$ is used as the main coordinate system associated with the Moon, the axes of which, in a different order, correspond to the above directions: the axis $C\xi_K$ is directed to the Earth ($C\xi_K = -C\xi$); the axis $C\zeta_K$ is the polar axis of inertia (axis $C\zeta_K = C\eta$); the axis $C\eta_K$ coincides with the axis $C\zeta$ ($C\eta_K = C\zeta$) and complements the coordinate system to the right.

Formally, let's rename the task parameters. For the coefficients of the second harmonic of the selenopotential, defined in the working coordinate system $O\xi\eta\zeta$, let's retain the usual notation, i.e., $J_2, C_{22}, S_{22}, C_{21}, S_{21}$, and for the same coefficients defined in the main coordinate system of the Moon $C\xi_K\eta_K\zeta_K$, we use the new notation (with the "K" index) J_2^K, C_{22}^K, S_{22}^K, C_{21}^K, S_{21}^K. It is the last coefficients obtained from satellite and laser observations. Thus, according to the latest observational data for the "Selenium" project, for the average values of these coefficients (for their constant components) their values were determined with high accuracy [5]. The relations between the selenopotential coefficients determined with respect to two coordinate systems are obtained above (1), (2).

As a result, the formulas for the librations of the Moon in variables are written as follows:

$$\delta l = \sum_{\|\mathbf{v}\|>0} \frac{\omega}{\omega_{\mathbf{v}}} S_{21}^{(\mathbf{v};K)} \cos\theta_{\mathbf{v}} = \sum_{\|\mathbf{v}\|>0} \frac{T_{\mathbf{v}}}{T} S_{21}^{(\mathbf{v};K)} \cos\theta_{\mathbf{v}}; \tag{21}$$

$$\delta g = \frac{2}{3J} \sum_{\|\mathbf{v}\|>0} \frac{\omega}{\omega_{\mathbf{v}}} J_2^{(\mathbf{v};K)} \sin\theta_{\mathbf{v}} = \frac{2}{3J} \sum_{\|\mathbf{v}\|>0} \frac{T_{\mathbf{v}}}{T} J_2^{(\mathbf{v};K)} \sin\theta_{\mathbf{v}}; \tag{22}$$

$$\delta\theta = \frac{1}{J} \sum_{\|\mathbf{v}\|>0} \frac{\omega}{\omega_{\mathbf{v}}} C_{21}^{(\mathbf{v};K)} \cos\theta_{\mathbf{v}} = \frac{1}{J} \sum_{\|\mathbf{v}\|>0} \frac{T_{\mathbf{v}}}{T} C_{21}^{(\mathbf{v};K)} \cos\theta_{\mathbf{v}}. \tag{23}$$

Here, $\omega_{\mathbf{v}} = v_1 n_M + v_2 n_S + v_3 n_F + v_4 n_D$; $T_{\mathbf{v}} = 2\pi / \omega_{\mathbf{v}}$. Formulas (21)–(23) are complemented by similar trigonometric representations for projections of angular velocity.

The work does not consider the effect of tidal variations in the selenopotential coefficients on the librations of the Moon. Due to the resonant nature of the rotation of the Moon, these perturbing factors are comparable in magnitude with those studied in this paper. In future work, they will be studied separately. For further research, the theory of physical libration of the Moon with a liquid core is of interest [7–9].

Variations of the projections of the vector of angular velocity on the axis of inertia of the body. Let's consider the general formulas connecting the projections of the angular velocity of rotation $\boldsymbol{\omega} = (p, q, r)$ and the projection of the angular momentum vector $\mathbf{G}$ of rotational motion,

$$G_\xi = G\sin\theta\sin l;\;\; G_\eta = G\sin\theta\cos l;\;\; G_\zeta = G\cos\theta \tag{24}$$

on the main (average or unperturbed) inertia axis of the deformable Moon [1]

$$\begin{aligned} p &= a\left(G_\xi - P\right) - f\left(G_\eta - Q\right) - e\left(G_\zeta - R\right);\\ q &= -f\left(G_\xi - P\right) + b\left(G_\eta - Q\right) - d\left(G_\zeta - R\right);\\ r &= -e\left(G_\xi - P\right) - d\left(G_\eta - Q\right) + c\left(G_\zeta - R\right), \end{aligned} \tag{25}$$

where coefficients $a,\ b, \ldots,\ f$ are known functions of time (11). In the general case, the equations include the projections of the kinetic moment of the Moon mantle particles relative displacements (due to tidal deformations as a result of the attraction of the Earth) onto the average axes of inertia: $P,\ Q$ and R. There are currently no data on these characteristics of the variable Moon, therefore we take them to be equal to zero.

Keeping only the terms of the first order of smallness with respect to the time variations of the components of the inertia tensor of the Moon (5) and the coefficients of the selenopotential (6), after the necessary transformations of equalities (25), the approximate formulas are obtained

$$\delta p = \frac{G}{A}\left(\cos\theta\sin l\delta\theta + \sin\theta\cos l\delta l - \frac{\delta A}{A}\sin\theta\sin l + \frac{\delta F}{B}\sin\theta\cos l + \frac{\delta E}{C}\cos\theta\right); \tag{26}$$

$$\delta q = \frac{G}{B}\left(\cos\theta\cos l\delta\theta - \sin\theta\sin l\delta l - \frac{\delta B}{B}\sin\theta\cos l + \frac{\delta F}{A}\sin\theta\sin l + \frac{\delta D}{C}\cos\theta\right); \tag{27}$$

$$\delta r = \frac{G}{C}\left(-\sin\theta\delta\theta - \frac{\delta C}{C}\cos\theta + \frac{\delta E}{A}\sin\theta\sin l + \frac{\delta D}{B}\sin\theta\cos l\right). \tag{28}$$

Here, the factors with variations of the Andoyer variables and variations of the components of the inertia tensor depend on the unperturbed values of the variables $\theta = \pi/2$ and $l = 0,$ therefore formulas (26)–(28) can be written in a simplified form

$$\delta p = \frac{G}{A}\left(\delta l + \frac{\delta F}{B}\right);\;\; \delta q = -\frac{G}{B}\frac{\delta B}{B};\;\; \delta r = \frac{G}{C}\left(-\delta\theta + \frac{\delta D}{B}\right), \tag{29}$$

or taking into account relations (5) in the accepted approximation

$$\delta p = \frac{G}{A}\left(\delta l + \frac{2}{J}\delta S_{22}\right); \ \delta q = -\frac{G}{3JB}\left(6\delta C_{22} - \delta J_2\right);$$
$$\delta r = \frac{G}{C}\left(-\delta\theta + \frac{1}{J}\delta S_{21}\right), \tag{30}$$

where variation δp corresponds to the projection of the angular velocity on the equatorial axis of inertia of the Moon $C\xi$, the variation δr corresponds to the projection of the angular velocity onto the other equatorial axis $C\zeta$, variation δq corresponds to variation in the axial rotation of the Moon and the duration of this rotation (LOD compared with similar characteristics for the Moon). Selenopotential coefficients differ from conventional (classical) notation. Again let's perform the predominance of the axes of inertia according to the rule given earlier, and introduce the classical notation of the projections of the angular velocity

$$\delta p_K = -\delta p; \ \delta q_K = \delta r; \ \delta r_K = \delta q, \tag{31}$$

also in accordance with the redesignation of the axes of inertia of the Moon used above. Then finally

$$\delta p_K = \frac{\omega}{J}\sum_{\|\mathbf{v}\|>0} C_{21}^{(\mathbf{v};K)} \sin\boldsymbol{\theta}_{\mathbf{v}};$$
$$\delta q_K = -\frac{\omega}{J}\sum_{\|\mathbf{v}\|>0} S_{21}^{(\mathbf{v};K)} \cos\boldsymbol{\theta}_{\mathbf{v}}; \tag{32}$$
$$\delta r_K = -\frac{2\omega}{3J}\sum_{\|\mathbf{v}\|>0} J_{2}^{(\mathbf{v};K)} \cos\boldsymbol{\theta}_{\mathbf{v}}.$$

The results of the analysis of the physical librations of the Moon caused by its tidal deformations for projections of angular velocity (32) and for variations of the three Andoyer variables (21)–(23) are given in Tables 1–3. It also shows the amplitudes $J_2^{(\mathbf{v};K)}$, $C_{21}^{(\mathbf{v};K)}$, $S_{21}^{(\mathbf{v};K)}$, the periods $T_{\mathbf{v}}$ and arguments $\theta_{\mathbf{v}}$ for the tidal variations of the second harmonic coefficients of the selenopotential. Table 1 shows the amplitudes, periods and arguments of the tidal variations of the lunar day duration, determined by analogy with the variations of the Earth day duration

$$\delta\left(LOD\right)_{\text{Moon}} = -T_F\,\frac{\delta r_K}{n_F}.$$

The amplitudes of the variations in the table 3 determined in milliseconds of time (ms).

Table 1

Forced librations of the Moon due to tidal deformations of the Moon, amplitudes and periods of cyclic variations of the projection of the angular velocity and the Andoyer variable

ν_1	ν_2	ν_3	ν_4	T_ν	$C_{21}^{(\nu)}$, 10^{-8}	p_ν, 1"·10^{-3}	θ_ν, 1"·10^{-3}
0	0	1	0	27.212	–2.2209	11.6421	–11.6421
1	0	1	0	13.691	–0.3028	1.5873	–3.1549
1	0	–1	0	2190.35	0.0613	–0.3213	–0.0040
1	0	–1	–2	14.666	0.0577	–0.3025	–0.0013
0	0	1	2	9.572	–0.0488	0.2558	–0.2538
2	0	1	0	9.146	–0.0355	0.1861	–0.5537
0	0	1	–2	32.281	0.0329	–0.1725	0.1738
1	0	1	2	7.104	–0.0106	0.0556	–0.1100
1	0	1	–2	188.201	–0.0095	0.0498	–0.0994
2	0	–1	0	27.906	0.0052	–0.0273	0.0266
0	0	3	0	9.071	–0.0037	0.0194	–0.0582
0	1	–1	–2	9.829	0.0032	–0.0168	–0.0154
3	0	1	0	6.867	–0.0032	0.0168	–0.0665
2	0	1	–2	24.036	0.0020	–0.0105	0.0313
2	0	1	2	5.648	–0.0014	0.0073	–0.0218

Table 2

Tidal variations of the axial rotation of the Moon δq (mas) and variations of the Andoye angular variable g (mas)

l_m	l_s	F	D	T_ν	J_2	δq_K, 1"·10^{-3}	δg, 1"·10^{-3}
0	0	0	0	0	9.446	33.0111	0.0000
1	0	0	0	27.555	1.5453	5.4004	7.9999
1	0	0	–2	31.812	0.2954	1.0323	1.5417
0	0	0	2	14.765	0.2584	0.9030	–0.0108
0	0	0	2	13.606	0.1933	0.6755	–0.0081
2	0	0	0	13.777	0.1267	0.4428	1.3118
1	0	0	2	9.614	0.041	0.1433	0.2105
1	0	2	0	9.108	0.037	0.1293	0.5795
2	0	0	–2	205.892	0.0137	0.0479	0.1424
3	0	0	0	9.185	0.0102	0.0356	0.1584
1	–1	0	0	29.803	0.0095	0.0332	0.0455
0	0	2	–2	173.31	–0.0094	–0.0329	–0.0989
1	0	0	–4	10.085	0.008	0.0280	0.0421
1	1	0	0	25.622	–0.0079	–0.0276	–0.0440
0	0	2	2	7.081	0.0059	0.0206	0.0616
2	0	2	0	6.846	0.0049	0.0171	0.1021

Table 3

Tidal variations of the axial rotation of the Moon δr_K (in milliseconds of arc) and variations of the Andoyer angular variable δl (in milliseconds of arc)

l_m	l_s	F	D	T_v	δS_{21}	δq_K, 1"·10^{-3}	K_2	δl, 1"·10^{-3}
1	0	−1	0	2190.35	−0.1227	0.6432	−0.01242	0.0080
1	0	1	0	13.691	0.1204	−0.6311	1.987576	1.2544
1	0	1	−2	188.201	−0.0283	0.1484	1.995582	−0.2960
2	0	1	0	9.146	0.0205	−0.1075	2.975153	0.3197
0	0	1	2	9.572	0.0188	−0.0986	0.991995	0.0978
0	0	1	−2	32.281	−0.0141	0.0739	1.008005	−0.0745
0	0	1	0	27.212	−0.0105	0.0550	1	−0.0550
2	0	−1	0	27.906	−0.0076	0.0398	0.975153	−0.0388
1	0	1	2	7.104	0.0058	−0.0304	1.979571	0.0602
2	0	1	−2	24.036	−0.0042	0.0220	2.983158	−0.0657
0	0	3	0	9.071	−0.0037	0.0194	3	−0.0582
3	0	1	0	6.867	0.0032	−0.0168	3.962729	0.0665
0	1	1	0	25.325	−0.0027	0.0142	1.074501	−0.0152
1	−1	1	0	13.197	−0.0016	0.0084	1.913075	−0.0160
2	0	1	2	5.648	0.0014	−0.0073	2.967147	0.0218
1	1	2	1	6.838	−0.001	0.0052	3.058075	−0.0160

Conclusion. Tidal deformations caused by the gravitational influence of the Earth lead to sensitive physical librations of the Moon with amplitudes of the order of a few milliseconds of arc. Variations in the duration of the day due to tidal deformations of the Moon are also significant and can reach tens of milliseconds. The obtained results should be taken into account when implementing lunar projects in the near future, in particular in the Japanese *ILOM* project for installing a telescope on the lunar surface in order to determine its orientation accurately [10]. High-precision measurements of the orientation of the Moon and its temporal changes open up new possibilities for reseaching in the internal structure and internal dynamics of the Moon, its deformations.

Translated by E. Ovsyannikova

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Please cite this article as:

Barkin M.Yu., Shkapov P.M., Hanada Hideo. The physical librations of the Moon caused by its tidal deformations. *Herald of the Bauman Moscow State Technical University, Series Natural Sciences*, 2019, no. 2, pp. 4–16.
DOI: 10.18698/1812-3368-2019-2-4-16