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BOUNDARY VALUE PROBLEMS NUMERICAL SOLUTION ON MULTIBLOCK GRIDS

S.I. Martynenko[1, 2] martynenko@ciam.ru
A.Yu. Varaksin[2, 3] varaksin_a@bmstu.ru

[1] **Central Institute of Aviation Motors, Moscow, Russian Federation**
[2] **Bauman Moscow State Technical University, Moscow, Russian Federation**
[3] **Joint Institute for High Temperatures, Russian Academy of Sciences, Moscow, Russian Federation**

Abstract

Results of theoretical analysis of the geometric multigrid algorithms convergence are presented for solving the linear boundary value problems on a two-block grid. In this case, initial domain could be represented as a union of intersecting subdomains, in each of them a structured grid could be constructed generating a hierarchy of coarse grids. Multigrid iteration matrix is obtained using the damped nonsymmetric iterative method as a smoother. The multigrid algorithm contains a new problem-dependent component — correction interpolation between grid blocks. Smoothing property for the damped nonsymmetric iterative method and convergence of the robust multigrid technique are proved. Estimation of the multigrid iteration matrix norm is obtained (sufficient convergence condition). It is shown that the number of multigrid iterations does not depend on either the step or the number of grid blocks, if interpolation of the correction between grid blocks is sufficiently accurate. Results of computational experiments are presented on solving the three-dimensional Dirichlet boundary value problem for the Poisson equation illustrating the theoretical analysis. Results obtained could be easily generalized to multiblock grids. The work is of interest for developers of highly efficient algorithms for solving the (initial-) boundary value problems describing physical and chemical processes in complex geometry domains

Keywords

Boundary value problems, multigrid methods, multiblock grids

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Introduction. Mathematical models of physicochemical processes (hydrodynamics, thermal conductivity, convective and radiation heat transfer, turbulent transfer, chemical kinetics, electromagnetism, elasticity theory problems, etc.)

in continua mechanics are usually the (initial-) boundary value problems for systems of nonlinear partial differential equations. Nonlinear boundary value problem approximation on a computational grid leads to a system of high-order nonlinear algebraic equations, which numerical solution requires enormous computational costs. In this regard, elaboration of effective methods for numerical solution of such boundary value problems is of great theoretical and application importance. The problems of numerical simulation associated with elaboration and theoretical justification of robust, efficient and parallel algorithms do not disappear by themselves, as more and more powerful and inexpensive computers appear. At least, this is due to two reasons: 1) complication of tasks put forward by both practice and theory; 2) need to carry out a large number of computational experiment series for to ensure sufficiently comprehensive study of an object [1].

In the mid-1980s, multigrid methods, which history started with pioneering works by outstanding Russian mathematicians R.P. Fedorenko [2, 3], N.S. Bakhvalov [4] and G.P. Astrakhantsev [5], are widely used in solving the application problems. In addition, personal computers appeared powerful enough to perform scientific and technical calculations. Computing technology gained wide access to engineers, physicists, chemists and specialists in other problem areas, who were not provided with sufficient training in computational mathematics, but faced the necessity to solve application problems requiring large computational capacity. In this regard, numerous attempts have been made since the mid-1980s to develop computational algorithms for programs based on the black-box software principle. By this time, disadvantages of the classical multigrid method (CMM) were well known and include the following difficulties:

1) *formalizing computations*: in fact, CMM is a set of problem-dependent components (smoothing procedure, type of coarse grids, method of specifying an operator on coarse grids, transfer operators, multigrid cycles, etc.), which optimal selection for the boundary value problem to be solved determines the optimal convergence rate of the given multigrid method[*]. It is obvious that such algorithms could hardly be used in the black-box programs;

2) *parallelizing computations*: it is necessary in parallel CMMs to distribute different amounts of computational work evenly between the same number of independent computers. When smoothing on coarse grids, amount of computational work decreases leading to an increase in the cost of data exchange [7]. In addition, several independent computers could stand idle, when coarse grids are being smoothed [7]. Therefore, efficiency of parallel CMMs is relatively low;

[*] However, the multigrid method could not be understood as a fixed algorithm [6].

3) *application in solving the boundary value problems in complex geometry domains*: using CMM implies that a computational grid makes it possible to build a hierarchy of coarse grids by enlarging the step. As a rule, these grids are difficult to build in domains with complex geometry.

Requirement to conduct scientific and technical calculations led to intensive development of the multigrid methods. As of today, it is rather hard to review the entire number of different CMM versions that were proposed, but only two of them are the most significant.

1. *Algebraic multigrid methods (AMM)*. Auxiliary systems of linear algebraic equations (SLAE) required to find the correction are built in these methods without involving geometric information about the computational grid [7, 8]. AMMs are efficient and highly formalized algorithms in solving the SLAEs obtained as a result of approximating the linear boundary value problems on unstructured grids. AMM is applied in the nonlinear case to solution of the SLAE obtained as a result of the discrete boundary value problem global linearization. In addition to the need for global linearization of the nonlinear algebraic equations initial system, problems are remaining associated with building the parallel AMMs.

2. *Robust Multigrid Technique (RMT)*. This technology is based on using the essential multigrid principle[**] in a single-grid algorithm to minimize the number of problem-dependent components [11, 12]. As applied to numerical solution of the boundary value problems for systems of nonlinear partial differential equations on structured grids, RMT consists of sequential application of the Seidel single-grid method with block ordering of unknowns (external iterations) and the Newton method (internal iterations) [10]. Algorithmic complexity of the Seidel method in the linear case is reduced to being close to optimal without involving the problem-dependent components. Smoother used in the RMT is the generalization of the Vanka iterative method and allows unified solution of a wide class of the boundary value problems, from problems for the Poisson equation to problems for the Navier — Stokes equations [13]. Using only a single grid for finding the correction led to problem-independent restriction and prolongation operators, uniform loading of independent computers in parallel execution, but slightly increased computational cost of the RMT iteration compared to the classical iteration [11, 12]. Although RMT is a single-

[**] The essential multigrid principle is to approximate the smooth (long wavelength) part of the error on coarser grids. The nonsmooth or rough part is reduced with a small number (independent of h) of iterations with a basic iterative method on the fine grid [9].

grid algorithm, it is more convenient to represent it as a version of geometric multigrid methods (GMM).

Computational domain geometry determines main features of the algorithm for numerical solution of the boundary value problems [11, 12]. Regular grid is assumed to be structured, if it generates a hierarchy of coarser grids. Three special cases are found in the applications:

1) *globally structured (single-block) grids*: hierarchy of the coarse grids is built within the entire area. Grids of this type could be built in domains with the simplest geometry;

2) *locally structured (multiblock) grids*: initial domain is represented as a union of intersecting subdomains, and in each of them a structured grid could be built generating a hierarchy of coarse grids. Grids of this type are often found in various applications;

3) *unstructured grids*: it is impossible to build a coarse grid hierarchy over the entire domain. In this case, a promising approach to building an effective algorithm for solving the nonlinear boundary value problems lies in the Auxiliary Space Method, which uses the auxiliary (structured) grid to find the correction [14, 15].

Purpose of this work is to theoretically study the GMM and RMT convergence on locally structured (multiblock) grids within the framework of the classical multigrid analysis proposed by W. Hackbusch [16].

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Smoothing on two-block grids. Let us assume that the Ω domain is representable as the union of subdomains Ω_k, and in each of them it is possible to build a structured grid G_k^h, but their $G^h = \bigcup_{k=1}^{K} G_k^h$ union is not a globally structured grid. Grids of this type will be called the locally structured or multiblock grids.

Next, let us consider the simplest case $K = 2$, the typical example is shown in Fig. 1: the Ω domain consists of two subdomains Ω_1 and Ω_2, in each of them a uniform structured grid is built.

Let us consider a linear boundary value problem:

$$L_{\Omega_1} g(x) = f_{\Omega_1}(x), \; x \in \Omega_1, \tag{1a}$$

$$L_{\Omega_2} g(x) = f_{\Omega_2}(x), \; x \in \Omega_2. \tag{1b}$$

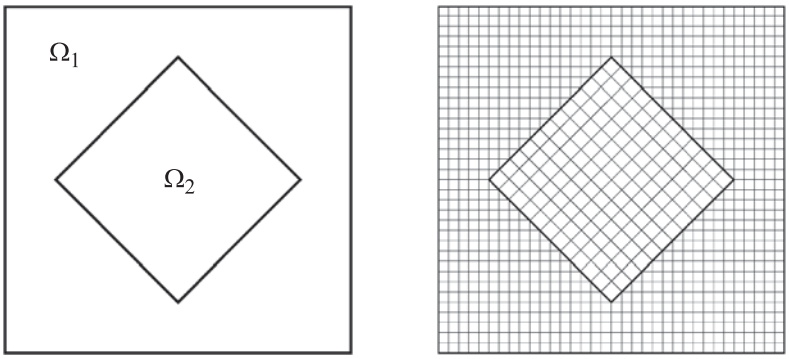

Fig. 1. $\Omega = \Omega_1 \cup \Omega_2$ domain (*a*) and locally structured (two-block) grid (*b*)

Here L_{Ω_1}, L_{Ω_2} are the linear elliptic differential operators defined in the subdomains Ω_1 and Ω_2; g is the required function; $x = (x_1, ..., x_d)^T$; f_{Ω_1}, f_{Ω_2} are the known functions in Ω_1 and Ω_2; $\Omega \in R^d$ is the given open domain with boundary $\partial\Omega$. The following boundary conditions are set on the $\partial\Omega$ outer boundary of the Ω domain:

$$L_{\partial\Omega_1} g(x) = f_{\partial\Omega_1}(x), \quad x \in \partial\Omega_1, \tag{1c}$$

and conjugation conditions are set on the $\partial\Omega_2$ inner boundary:

$$L_{\partial\Omega_2} g(x) = f_{\partial\Omega_2}(x), \quad x \in \partial\Omega_2. \tag{1d}$$

Let us construct in each of the Ω_1 and Ω_2 the G_1 and G_2 with steps h_1 and h_2. Approximation of the (1a)–(1c) original differential problem on the G_1 grid leads to a SLAE of the following form:

$$Bu + \widehat{B}\widehat{u} = b_u,$$

where B is the matrix being a grid analogue of operator L_{Ω_1}; $\widehat{u}$ is the value of the u sought grid function in the G_1, grid nodes located outside the Ω_1 domain (i.e., in the Ω_2 domain). Using interpolation, let us express $\widehat{B}\widehat{u}$ through the υ function values in the real nodes of grid G_2:

$$\widehat{B}\widehat{u} = C\upsilon.$$

The C matrix form depends on the grid blocks intersection type (common boundary or intersection of blocks), method and order of interpolation (node to node, as shown in Fig. 1, or interpolation of different order). Then the resulting SLAE takes the following form:

$$Bu + C\upsilon = b_u.$$

Similarly, approximation of the (1b)–(1d) initial differential problem on the G_2 grid leads to a SLAE of the following form:

$$Du + F\upsilon = b_\vartheta.$$

Here F is a matrix being the grid analogue of operator L_{Ω_2}; Du are the terms obtained as a result of excluding the υ desired grid function values in the fictitious nodes of grid G_2, located outside the Ω_2 domain (i.e., in the Ω_1 domain).

Error of interblock interpolation in approximations to solution should be not less than the error of approximation with operators L_{Ω_1} and L_{Ω_2}. Interblock interpolation operator design is determined by the block intersection type, required accuracy and smoothness of the differential operator coefficients.

Thus, approximation of the initial differential problem (1) on the two-block grid formed by grids G_1 and G_2, leads to a SLAE of the following form:

$$\begin{pmatrix} B & C \\ D & F \end{pmatrix}\begin{pmatrix} u \\ \upsilon \end{pmatrix} = \begin{pmatrix} b_u \\ b_\vartheta \end{pmatrix}, \tag{2}$$

where u, υ are discrete analogs of the g function on grids G_1 and G_2. The B and C invertible matrices are the grid analogs of operators L_{Ω_1} and L_{Ω_2}, while the C and D matrices are generally rectangular; what is more, $C^T \neq D$. The C and D ensure interconnection of the grid problems on grids G_1 and G_2.

Let us consider the simplest iterative method for solving the SLAE (2):

$$W_B\left(u^{(n+1)} - u^{(n)}\right) = b_u - Bu^{(n)} - C\upsilon^{(n)}, \tag{3a}$$

$$W_F\left(\upsilon^{(n+1)} - \upsilon^{(n)}\right) = b_\upsilon - Du^{(n+1)} - F\upsilon^{(n)}, \tag{3b}$$

where W_B, W_F are the splitting matrices for the B and F matrices.

Substituting $u^{(n+1)}$ from (3a) into (3b), the following is obtained:

$$\begin{pmatrix} u^{(n+1)} \\ \upsilon^{(n+1)} \end{pmatrix} = \begin{pmatrix} S_B & -W_B^{-1}C \\ -W_F^{-1}DS_B & \tilde{S}_F \end{pmatrix} + \begin{pmatrix} W_B^{-1}b_u \\ W_F^{-1}b_\upsilon - W_F^{-1}DW_B^{-1}b_u \end{pmatrix}, \tag{4}$$

where $\tilde{S}_F = I - W_F^{-1}\left(F - DW_B^{-1}C\right)$; $S_B = I - W_B^{-1}B$.

Equation (4) could be written down in the canonical form:

$$\psi^{(n+1)} = \left(I - W^{-1}\tilde{A}\right)\psi^{(n)} + W^{-1}\tilde{b}, \tag{5}$$

where

$$\psi = \begin{pmatrix} u \\ \upsilon \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} B & C \\ D & F \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & DW_B^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ B & C \end{pmatrix},$$

$$W = \begin{pmatrix} W_B & 0 \\ 0 & W_F \end{pmatrix}, \quad \tilde{b} = \begin{pmatrix} I & 0 \\ -DW_B^{-1} & I \end{pmatrix} \begin{pmatrix} b_u \\ b_\upsilon \end{pmatrix}.$$

Classical analysis of the multigrid methods convergence is based on the smoothing and approximation properties [16]. Unfortunately, the most studied is the symmetric case [17]. In the nonsymmetric case, results were obtained only for a smoother with parameter [11, 12]. Let us write down (5), as follows:

$$\psi^{(\nu+1)} = S(\omega)\psi^{(\nu)} + W^{-1}\tilde{b}, \tag{6}$$

where $S(\omega)$ is the matrix of smoothing iterations,

$$S(\omega) = I - \frac{1}{1+\omega} W^{-1}\tilde{A};$$

$\omega \geq 0$ is a certain parameter.

Proof of smoothing properties in the asymmetric case is based on the following lemmas.

Lemma 1. *Let the* $\|Q\| \leq 1$ *be performed for the* $Q \in \mathbb{R}^{n\times n}$ *matrix in a certain operator norm. Then, the following is correct in the same norm:*

$$\frac{1}{\left(1+\omega^{\nu+1}\right)} \left\| (I-Q)(\omega I + Q)^{\nu} \right\| \leq \frac{1}{\sqrt{e\omega\nu}}, \ \nu = 1, 2, ..., \tag{7}$$

if $3 - 2\sqrt{2} \leq \omega \leq 3 + 2\sqrt{2}$.

Proof of Lemma 1 is provided in [11, 12]. Then, the following lemma on smoothing properties is correct.

Lemma 2. *Let the* $\|S(0)\| < 1$ *and* $\|W\| \leq C\|\tilde{A}\|$ *be performed for the* $S(0) \in \mathbb{R}^{n\times n}$ *matrix of smoothing iterations in a certain operator norm, and C is a certain constant. Then, in the same norm at* $3 - 2\sqrt{2} \leq \omega \leq 3 + 2\sqrt{2}$ *the following is correct:*

a)
$$\left\| \tilde{A} S^{\nu}(\omega) \right\| \leq C \frac{1+\omega}{\sqrt{e\omega\nu}} \|\tilde{A}\|, \tag{8}$$

b)
$$\left\| r^{(\nu)} \right\| \leq C \frac{1+\omega}{\sqrt{e\omega\nu}} \|\tilde{A}\| \left\| \psi^{(0)} - \psi \right\| \leq C \frac{1+\omega}{\sqrt{e\omega\nu}} \operatorname{cond}(\tilde{A}) \left\| r^{(0)} \right\|, \tag{9}$$

where $r^{(\nu)} = \tilde{b} - \tilde{A}\psi^{(\nu)}$ *is the discrepancy of the ν-th approximation to solution* $\psi = \tilde{A}^{-1}\tilde{b}$.

◄ Let us prove item a). Taking into account (7), it is easy to obtain:

$$\left\|\tilde{A}S^{\nu}(\omega)\right\| \le \frac{1}{(1+\omega)^{\nu+1}}\left\|(I-S(0))(\omega I+S(0))^{\nu}\right\|(1+\omega)\|W\| \le \frac{1+\omega}{\sqrt{e\omega\nu}}\|W\|,$$

whence, (8) follows.

Let us prove item b). For the exact solution (6), the following is true: $\psi^{(\nu)} - \psi = S^{\nu}(\omega)\left(\psi^{(0)} - \psi\right)$. Whence, multiplying by the $\tilde{A}$ matrix, it follows:

$$-r^{(\nu)} = \tilde{A}S^{\nu}(\omega)\left(\psi^{(0)} - \psi\right).$$

Further, taking into account (8), it is easy to obtain (9). ►

Nevertheless, traditional approach to proving the smoothing property in the nonsymmetric case based on estimates like (7), is rather crude: for example, the nonsymmetric Seidel method has the smoothing property, which does not follow from (8) for ω = 0. In this regard, additional studies are required in case $\omega \to 0$.

This approach could easily be generalized in case $K > 2$, i.e., if the number of grid blocks is more than two.

Multigrid iterations on two-block grids. It is easier to analyze the RMT single-grid convergence under an assumption that smoothing is carried out on a multigrid structure, i.e., a special sequence of subgrids in the finest grid [11, 12]. In this case, multigrid structure is used only for approximating the Σ-modified boundary value problem (1) by the finite volume method. Then, the difference boundary value problem could be put down in the matrix form:

$$A_l c_l = R_{0\to l}\, r_0, \quad l = 0,1,...,L_3^+, \tag{10}$$

where A_l is the coefficient matrix,

$$A_l = \begin{pmatrix} B_l & C_l \\ D_l & F_l \end{pmatrix}.$$

B_l, C_l, D_l and F_l matrices are having the block-diagonal shape, and the number of blocks along the main diagonal is equal to the number of grids forming the given grid level 3^{dl}, $d = 2,3$, $l = 0,1,...,L_3^+$, where L_3^+ is the grid level number consisting of the most coarse grids built by tripling the step. Vectors of the unknown c_l and r_0 have the following form:

$$c_l = \begin{pmatrix} c_l^u \\ c_l^\upsilon \end{pmatrix}, \quad r_0 = \begin{pmatrix} r_0^u \\ r_0^\upsilon \end{pmatrix} = \begin{pmatrix} b_u - B_0\hat{u}^{(q)} - C_0\hat{\upsilon}^{(q)} \\ b_\upsilon - D_0\hat{u}^{(q)} - F_0\hat{\upsilon}^{(q)} \end{pmatrix}.$$

Here c_0^u, c_0^υ are corrections to approximation in the $\hat{u}$ and $\hat{\upsilon}$ solution in grids G_1 and G_2; q is the multigrid iteration number. The l subscript indicates belonging to the l grid level, with $l = 0$ being the finest grid. Restriction operator

$$R_{0\to l} = \begin{pmatrix} R_{0\to l}^{G_1} & 0 \\ 0 & R_{0\to l}^{G_2} \end{pmatrix}$$

projects the r_0 fastening vector of residuals from the G_1 and G_2 smallest grids ($l = 0$) onto grids of level $l = 0, 1, ..., L_3^+$. It is obvious that $R_{0\to 0} = I$. Unlike GMM, the RMT contraction operator is not depending on the problem being solved, and its design is evidently determined by the additivity property of a definite integral with respect to subdomains [11, 12].

Smoothing iterations (5) on a multigrid structure generated by the two-block grid could be written down as:

$$c_l^{(\nu_l+1)} = S_l\left(\omega_l\right)c_l^{(\nu_l)} + W_l^{-1}R_{0\to l}^{*}r_0, \tag{11}$$

where

$$S_l\left(\omega_l\right) = I - \frac{1}{1+\omega_l}W_l^{-1}\tilde{A}_l, \qquad \tilde{A}_l = \begin{pmatrix} B_l & C_l \\ D_l & F_l \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & D_lW_{B_l}^{-1} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ B_l & C_l \end{pmatrix},$$

$$W_l = \begin{pmatrix} W_{B_l} & 0 \\ 0 & W_{F_l} \end{pmatrix}; \qquad R_{0\to l}^{*} = \begin{pmatrix} I & 0 \\ -D_lW_{B_l}^{-1} & I \end{pmatrix}\begin{pmatrix} R_{0\to l}^{G_1} & 0 \\ 0 & R_{0\to l}^{G_2} \end{pmatrix}.$$

RMT uses the sawtooth cycle (i.e., the V-cycle without preliminary smoothing [9]). After smoothing is carried out on all grids of the $l + 1$ level, transition to finer grids of the l level is performed:

$$c_l^{(0)} = P_{l+1\to l}c_l^{(\nu_l+1)},$$

where $P_{l+1\to l}$ is the RMT prolongation problem-independent operator (permutation matrix), i.e., prolonged correction value from the adjacent level with $l + 1$ coarser grids is accepted as the initial approximation for smoothing iterations on the l level grids l [11, 12]. At the L_3^+ level with the coarsest grids, SLAEs (10) are solved exactly in the same way.

Hence, it is easy to obtain the following using [11]:

$$c_l - c_l^{(0)} = P_{l+1\to l}\left(c_l - c_l^{(\nu_l+1)}\right) + d_l R^*_{0\to l}\, r_0,$$

where $c_l = \tilde{A}_l^{-1} R_{0\to l}\, r_0$ is the exact correction value; d_l is the matrix and

$$d_l = \tilde{A}_l^{-1} - P_{l+1\to l}\, \tilde{A}_{l+1}^{-1} + d_l R^*_{l\to l+1}. \tag{12}$$

Examining the difference between correction exact and approximate values at levels $l = L_3^+, ..., 1, 0,$ the following is obtained:

$$c_l - c_l^{(\nu_l)} = Q_l\left(R^*_{0\to l}\, r_0 - \tilde{A}_l c_l\right),$$

where Q_l is the matrix determined in a recurrent manner,

$$Q_l = \begin{cases} S_l^{\nu_l}\left(\omega_l\right)\left(d_l R^*_{0\to l} + P_{l+1\to l}\, Q_{l+1}\right), & l = 0,1,2,..., L_3^+ - 2; \\ S_l^{\nu_l}\left(\omega_l\right) d_l R^*_{0\to l}, \quad & l = L_3^+ - 1. \end{cases} \tag{13}$$

Correction value after the RMT multigrid iteration would be:

$$c_0^{(\nu_0)} = c_0 - Q_0 r_0 = \left(\tilde{A}_0^{-1} - Q_0\right)\left(R^*_{0\to l}\, r_0 - \tilde{A}_l c_l\right).$$

Then, it is easy to obtain a new approximation to solution:

$$\psi^{(q+1)} = \psi^{(q)} + c_0^{(\nu_0)} = Q_0 \tilde{A}_0 \psi^{(q)} + \left(\tilde{A}_0^{-1} - Q_0\right) R^*_{0\to l}\, r_0, \tag{14}$$

i.e., the multigrid iteration matrix on a multigrid structure generated by the two-block grid is $Q_0 \tilde{A}_0$.

Classical analysis of the multigrid methods convergence is based on the following statements [17].

Statement 1. *Smoothing property: there is the* $\eta(\nu_l)\colon \mathbb{R}_+ \to \mathbb{R}_+$ *function, such that* $\eta(\nu_l) \to 0$ *for* $\nu_l \to \infty$ *and*

$$\left\| \tilde{A}_l S_l^{\nu_l} \right\| \le \eta(\nu_l) \left\| \tilde{A}_l \right\|, \quad l = 0,1,...., L^+ - 1. \tag{15}$$

Statement 2. *Approximation property: for a certain constant* $C_A > 0$

$$\left\| d_l \right\| = \left\| \tilde{A}_l^{-1} - P_{l+1\to l}\, \tilde{A}_{l+1}^{-1} R^*_{l\to l+1} \right\| \le C_A \left\| \tilde{A}_l \right\|^{-1}, \quad l = 0,1..., L^+ - 1. \tag{16}$$

Let us note that from (8) it follows:

$$\eta(\nu_l) = C \frac{1+\omega_l}{\sqrt{e\omega_l \nu_l}}. \tag{17}$$

Let us rewrite (14) as:

$$r_0^{(q+1)} = \tilde{A}_0 Q_0 r_0^{(q)},$$

where

$$\tilde{A}_0 Q_0 = \tilde{A}_0 S_0^{\nu_0}(\omega_0) d_0 +$$

$$+ \sum_{l=1}^{L_3^+-1} \prod_{k=0}^{l-1} \tilde{A}_k S_k^{\nu_k}(\omega_k) P_{k+1\to k} \tilde{A}_{k+1}^{-1} \tilde{A}_l S_l^{\nu_l}(\omega_l) d_l R_{0\to l}^{\star}, \qquad (18)$$

d_l is the matrix given according to (12).

The following theorem on the RMT convergence is correct.

Theorem. *Supposing that smoothing (15) and approximation (16) properties are satisfied, as well as* $\left\| I - W_l^{-1}\tilde{A}_l \right\| < 1$, $\left\| R_{0\to l+1}^{\star} \right\| \le C_R$ *and* $3 - 2\sqrt{2} \le \omega_l \le 3 + 2\sqrt{2}$. *Then, the RMT multigrid iterations converge, and*

$$\left\| \tilde{A}_0 Q_0 \right\| \le C_A C C_R \sum_{l=0}^{L_3^+-1} C_{\star}^l \frac{1+\omega_l}{\sqrt{e\omega_l \nu_l}}. \qquad (19)$$

◄ From (18) follows the estimate:

$$\left\| \tilde{A}_0 Q_0 \right\| \le \left\| \tilde{A}_0 S_0^{\nu_0}(\omega_0) d_0 \right\| +$$

$$+ \sum_{l=1}^{L_3^+-1} \prod_{k=0}^{l-1} \left\| \tilde{A}_k S_k^{\nu_k}(\omega_k) P_{k+1\to k} \tilde{A}_{k+1}^{-1} \right\| \left\| \tilde{A}_l S_l^{\nu_l}(\omega_l) d_l \right\| \left\| R_{0\to l}^{\star} \right\|. \qquad (20)$$

Using smoothing (15), (17) and approximation (16) properties, the following is obtained:

$$\left\| \tilde{A}_l S_l^{\nu_l}(\omega_l) d_l \right\| \le \left\| \tilde{A}_l S_l^{\nu_l}(\omega_l) \right\| \left\| d_l \right\| \le C_A C \frac{1+\omega_l}{\sqrt{e\omega_l \nu_l}}, \; l = 0, 1, 2, ..., \; L_3^+ - 1. \quad (21)$$

Further, the following inequality is evident:

$$\left\| \tilde{A}_k S_k^{\nu_k}(\omega_k) P_{k+1\to k} \tilde{A}_{k+1}^{-1} \right\| \le C_{\star} \left\| \tilde{A}_k S_k^{\nu_k}(\omega_k) \tilde{A}_k^{-1} \right\| \le C_{\star}, \qquad (22)$$

where $C_{\star}$ is a certain constant. Then, estimate (20) taking into account (21) and (22) takes the form (19). ►

Theorem proved shows that RMT convergence does not depend on the step of a two-block grid, but is determined only by the number of the ν_l, smoothing iterations performed on the l level grids. Implementing sufficient number of smoothing iterations makes it possible to achieve $\left\| \tilde{A}_0 Q_0 \right\| < 1$ (sufficient

convergence condition). Results obtained could be easily generalized to other GMM versions and to multiblock grids with more than two blocks.

Let us analyze the RMT labor content in solving a system consisting of N_M linear differential equations on the multiblock grid G. Let the G grid consist of blocks G_k, $k = 1, 2, ..., K$, and each of them contains the N_{G_k} nodes. Seidel method is selected as the smoothing procedure with special block ordering of unknowns (Vanka-type smoother [13]), i.e., the system of differential equations is solved jointly. Let us assume that computational effort to interpolate approximation to solution between the grid blocks is much less than computational effort to perform a single iteration according to the Vanka method. Labor content of a single iteration by the Vanka method would constitute the $W_{(1)} = Cn_b^{-2}(N_G N_M)^3$ arithmetic operations, where C is a certain constant; n_b is the number of unknowns in block ($1 \ll n_b \le N_G N_M$); N_G is the number of nodes:

$$N_G = \sum_{k=1}^{K} N_{G_k}.$$

Then, total labor content of solving the system would constitute $W = Cq\nu n_b^{-2}(N_G N_M)^3(L+1)$ arithmetic operations, where q is the number of multigrid iterations (not depending on $N_G N_M$); ν is the number of Vanka smoothing iterations on a multigrid structure; $L+1 \sim \lg \max\limits_{1 \le k \le K} N_{G_k} < \lg N_G$.

Then, labor content of solving a system of linear differential equations on the G multiblock grid would constitute $W = \overline{C}q\nu n_b^{-2}(N_G N_M)^3 \lg N_G$ arithmetic operations. In case of a single equation ($N_M = 1$) and using the pointwise ordering of unknowns, the $W = \overline{C}q\nu N_G \lg N_G$ arithmetic operations are obtained. Note that in a 3D case, labor content in the single-grid Seidel method with the pointwise ordering of unknowns would constitute $W = CN_G^{5/3}$ arithmetic operations.

Computational experiments. 3D Dirichlet problem for the Poisson equation was chosen as the model task

$$\Delta w = -f$$

in the Ω single cube. Exact solution is as follows:

$$w_e(x, y, z) = \exp(x + y + z).$$

This solution defines the f right-hand side and the boundary conditions.

Let us construct in the Ω domain a uniform grid G. Difference analogue of the initial boundary value problem obtained by approximation on a seven-point template could be written in the following matrix form:

$$A\omega^h = b^h.$$

The first test is intended for numerical solution of the model boundary value problem in the Ω unit cube (i.e., without dividing a domain into subdomains). After each q multigrid iteration, the $R = \left\| b^h - A\omega^h \right\|_\infty$ residual vector norm and the $E = \max \left| \omega_e \left(x_i, y_j, z_k \right) - \omega_{ijk}^h \right|$ numerical solution error are calculated. Stopping criterion for the multigrid iterations is taken as $R < 10^{-6}$.

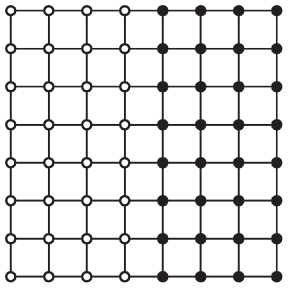

Fig. 2. 2D two-block grid (o — G_1 block; • — G_2 block)

Results of the first test are used to illustrate possible decrease in the convergence rate of the RMT multigrid iterations caused by dividing the Ω domain in two subdomains.

The second test is intended for numerical solution of the model boundary value problem on a two-block grid built in the Ω unit cube. Uniform grid G is divided in two blocks G_1 and G_2 along plane $x = 0.5$. Example of splitting a uniform mesh in the 2D case is presented in Fig. 2. After each q multigrid iteration, the $R_{21} = \left\| b^h - \tilde{A}\omega^h \right\|_\infty$ residual vector norm and the $E_{21} = \max\limits_{ijk} \left| \omega_e \left(x_i, y_j, z_k \right) - \omega_{ijk}^h \right|$ numerical solution error on the G_1 grid were calculated, as well as $R_{22} = \left\| b^h - \tilde{A}\omega^h \right\|_\infty$ and $E_{22} = \max\limits_{ijk} \left| \omega_e \left(x_i, y_j, z_k \right) - \omega_{ijk}^h \right|$ were calculated on grid G_2. Stopping criterion for the multigrid iterations is taken as $\max \left(R_{21};\, R_{22} \right) < 10^{-6}$. Smoothing is carried out alternately on the subgrids of grids G_1 and G_2.

The third test differs from the second only in smoothing: at first, iteration smoothing is performed on G_1, then on G_2. After each q multigrid iteration, the $R_{31} = \left\| b^h - \tilde{A}\omega^h \right\|_\infty$ residual vector norm and the $E_{31} = \max\limits_{ijk} \left| \omega_e \left(x_i, y_j, z_k \right) - \omega_{ijk}^h \right|$ numerical solution error are calculated on the G_1 grid, as well as $R_{32} = \left\| b^h - \tilde{A}\omega^h \right\|_\infty$ and $E_{32} = \max\limits_{ijk} \left| \omega_e \left(x_i, y_j, z_k \right) - \omega_{ijk}^h \right|$ on grid G_2. Stopping criterion for the multigrid iterations is taken as $\max \left(R_{31};\, R_{32} \right) < 10^{-6}$.

Seidel method with ordering the unknowns into blocks $3\times3\times3$ was selected as a smoother. Calculations were carried out on grids $101\times101\times101$ $\left(h_x = h_y = h_z = 1/100\right)$ and $301\times301\times301$ $\left(h_x = h_y = h_z = 1/300\right)$.

Results of the computational experiment are shown in Fig. 3. With alternating smoothing on the grid blocks (second test), RMT convergence on the two-block grid does not practically differ from convergence on the monoblock grid (first test). Sequential smoothing on each block leads to a slowdown in the RMT.

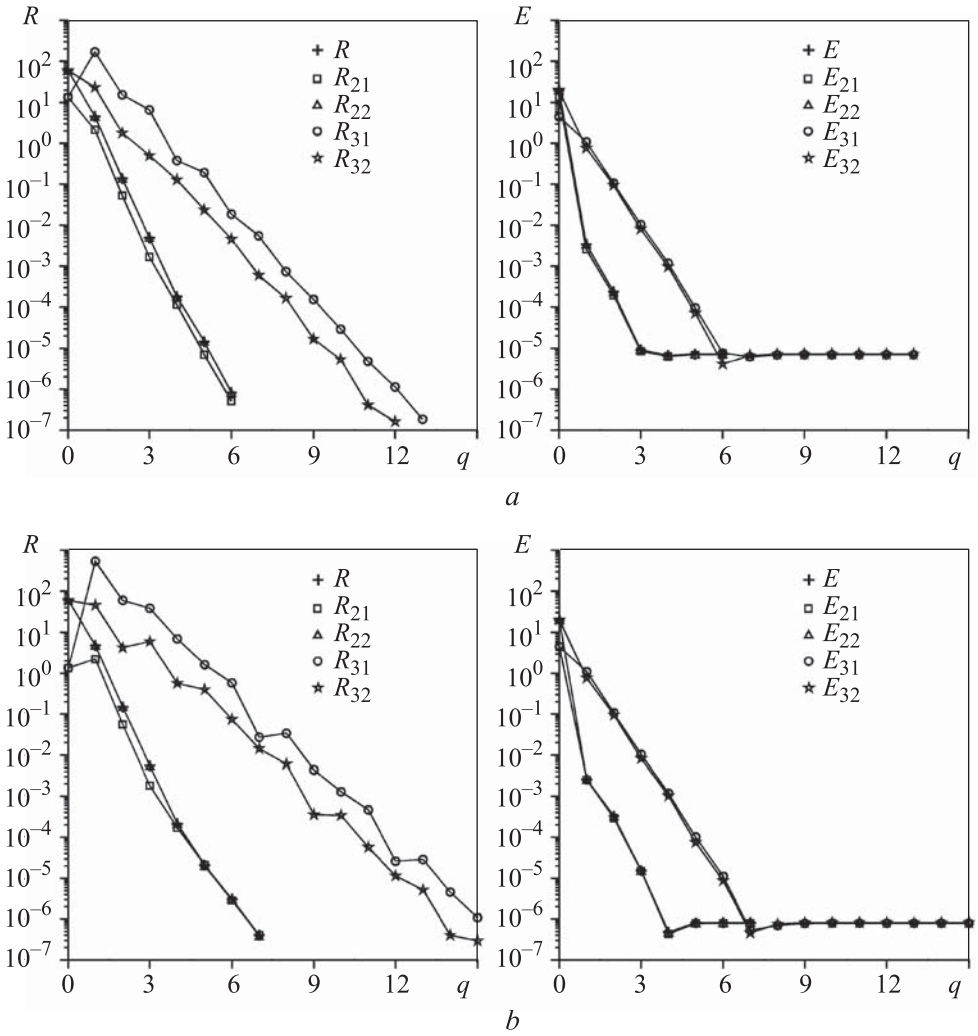

Fig. 3. RMT convergence on grids $101\times101\times101$ (*a*) and $301\times301\times301$ (*b*)

Conclusion. Results of the performed theoretical analysis and the executed computational experiments demonstrate that with alternating smoothing on grid blocks the number of multigrid iterations does not depend on either the

step size, or on the number of grid blocks. However, there still remain gaps in theoretical analysis of the multigrid methods convergence associated, in particular, with proving the smoothing property for nonsymmetric iterative methods using block ordering of the unknowns.

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Martynenko S.I. — Dr. Sc. (Phys.-Math.), Researcher, Department of Special Aviation Engines and Chemotology, Central Institute of Aviation Motors (Aviamotornaya ul. 2, Moscow, 111116 Russian Federation); Professor, Department of Gas Turbine Power Plants and Renewable Energy, Bauman Moscow State Technical University (2-ya Baumanskaya ul. 5, str. 1, Moscow, 105005 Russian Federation).

Varaksin A.Yu. — Corresponding Member of the Russian Academy of Sciences, Dr. Sc. (Phys.-Math.), Head of Department of Gas Turbine Power Plants and Renewable Energy, Bauman Moscow State Technical University (2-ya Baumanskaya ul. 5, str. 1, Moscow, 105005 Russian Federation); Head of Laboratory, Joint Institute for High Temperatures, Russian Academy of Sciences (Izhorskaya ul. 13, str. 2, Moscow, 125412 Russian Federation).

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